21.05.2025 (1)
Calculus Lect
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Standard basis

7.15 $v = (1,2) = (1,0) + (0,2) = (1,0) + 2(0,1)$
$= \hat{\imath} + 2\hat{\jmath}$

7.16 $u = (-1,3) = -\hat{\imath} + 3\hat{\jmath}$
$v = (-3,1,2) = -3\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and
$u = (-1,1,-2) = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$.

Proofs

7.11 Let $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
To show: $\|\lambda u\| = |\lambda| \cdot \|u\|$.
Write $u = (u_1, u_2, \ldots, u_n)$.

$$\begin{aligned}
LHS = \|\lambda u\| &= \|\lambda(u_1, u_2, \ldots, u_n)\| \\
&= \|(\lambda u_1, \lambda u_2, \ldots, \lambda u_n)\| \\
&= \sqrt{(\lambda u_1)^2 + (\lambda u_2)^2 + \cdots + (\lambda u_n)^2} \\
&= \sqrt{\lambda^2 u_1^2 + \lambda^2 u_2^2 + \cdots + \lambda^2 u_n^2} \\
&= \sqrt{\lambda^2 (u_1^2 + \cdots + u_n^2)} \\
&= \sqrt{\lambda^2}\sqrt{u_1^2 + \cdots + u_n^2} \\
&= |\lambda| \cdot \|u\| = RHS. //
\end{aligned}$$

7.13 Let $u \in \mathbb{R}^n$. Prove that $\frac{1}{\|u\|} u$ is a unit vector.

To show: $\| \frac{1}{\|u\|} u \| = 1$.

By 7.11,

$$\| \frac{1}{\|u\|} u \| = | \frac{1}{\|u\|} | \cdot \|u\|$$

$$= \frac{1}{\|u\|} \cdot \|u\| = 1.$$

//.

7.20 Let $u, v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Show that (a) $\lambda \cdot (u \bullet v) = (\lambda u) \bullet v = u \bullet (\lambda v)$.

(b) $u \bullet u = \|u\|^2$

Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

(a) To show: $\lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \langle u, \lambda v \rangle$.

$LHS = \lambda \langle u, v \rangle = \lambda \langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle$

$= \lambda (u_1 v_1 + \cdots + u_n v_n) = \lambda u_1 v_1 + \cdots + \lambda u_n v_n$.

$MHS = \langle \lambda u, v \rangle = \langle \lambda (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle$

$= \langle (\lambda u_1, \ldots, \lambda u_n), (v_1, \ldots, v_n) \rangle$

$= \lambda u_1 v_1 + \lambda u_2 v_2 + \cdots + \lambda u_n v_n$.

$RHS = \langle u, \lambda v \rangle = \langle (u_1, \ldots, u_n), \lambda (v_1, \ldots, v_n) \rangle$

$= \langle (u_1, \ldots, u_n), (\lambda v_1, \ldots, \lambda v_n) \rangle = u_1 \lambda v_1 + \cdots + u_n \lambda v_n$

$= \lambda u_1 v_1 + \cdots + \lambda u_n v_n.$

So LHS = MHS = RHS.

(b) To show: $\langle u, u\rangle = \|u\|^2$.

$LHS = \langle (u_1, \ldots, u_n), (u_1, \ldots, u_n)\rangle = u_1^2 + u_2^2 + \cdots + u_n^2.$

$RHS = \|u\|^2 = \left(\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\right)^2 = u_1^2 + u_2^2 + \cdots + u_n^2.$

So LHS = RHS.

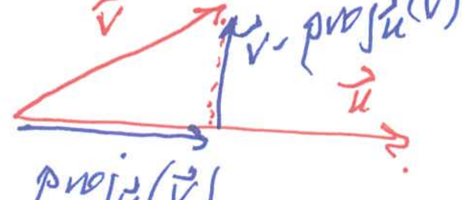
Angles and projections

The angle between u and v is the function $\theta: \mathbb{R}^n \times \mathbb{R}^n \to [0, \pi]$ given by

$$\theta(u, v) = \arccos\left(\frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}\right).$$

The projection of v onto u is the function $\mathrm{proj}_u : \mathbb{R}^n \to \mathbb{R}u$ given by

$$\mathrm{proj}_u(v) = \frac{1}{\|u\|^2}\langle u, v\rangle u$$



The lecture slides write

$$v_{\parallel} = (\hat{u} \cdot v)\hat{u} = \left\langle \frac{1}{\|u\|} u, v\right\rangle \frac{1}{\|u\|} u = \frac{1}{\|u\|^2}\langle u, v\rangle u$$
$$= \mathrm{proj}_u(v)$$

and $v_{\perp} = v - v_{\parallel} = v - \mathrm{proj}_u(v)$.

7.22 Let $u=(-2,1,2)$ and $v=(1,-1,0)$.
Then the angle between u and v is

$$\theta(u,v) = \theta((-2,1,2),(1,-1,0))$$

$$= \arccos\left(\frac{\langle(-2,1,2),(1,-1,0)\rangle}{\|(-2,1,2)\|\cdot\|(1,-1,0)\|}\right)$$

$$= \arccos\left(\frac{(-2)\cdot 1+1\cdot(-1)+2\cdot 0}{\sqrt{(-2)^2+1^2+2^2}\cdot\sqrt{1^2+(-1)^2+0^2}}\right)$$

$$= \arccos\left(\frac{-2-1+2}{\sqrt{9}\sqrt{2}}\right) = \arccos\left(\frac{-3}{3\sqrt{2}}\right)$$

$$= \arccos\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$$

7.23 Let $u=(3,1,-2)$ and $v=(1,0,5)$.
Then

$$\mathrm{proj}_u(v) = \frac{1}{\|u\|^2}\langle u,v\rangle u = \frac{1}{(\sqrt{3^2+1^2+2^2})^2}(3\cdot 1+1\cdot 0+(-2)(5))u$$

$$= \frac{1}{(\sqrt{14})^2}\cdot(-7)u = \frac{-1}{2}u = \frac{-1}{2}(3,1,-2)$$

$$= \left(-\frac{3}{2},\frac{-1}{2},1\right)$$

and

$$v-\mathrm{proj}_u(v) = (1,0,5)-\left(\frac{-3}{2},\frac{-1}{2},1\right) = \left(\frac{5}{2},\frac{1}{2},4\right).$$