16.05.2025
Calculus Lect.
A. Ram

Tree growth

$$h'(t) = a(1-b\,h(t)) \quad \text{with} \quad h(0)=0$$

and $a, b \in \mathbb{R}_{>0}$ is a model for tree height $h(t)$.

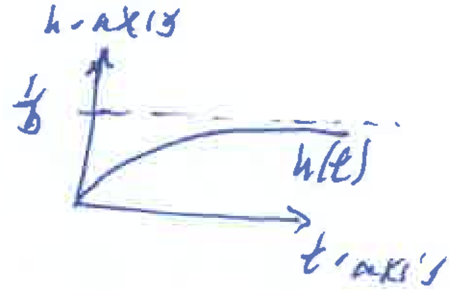
$$h'(0) = a(1-b\,h(0)) = a(1-0) = a$$

is the growth rate at birth.

If $\frac{dh}{dt} = 0$ then $a - ab\,h(t) = 0$

and $h(t) = \frac{a}{ab} = \frac{1}{b}$.

So the height of the tree stabilizes at a maximum height of $1/b$.



Claim: $h(t) = \frac{1}{b}(1 - e^{-abt})$.

Check: $\frac{dh}{dt} = \frac{1}{b}(0 - (-ab)e^{-abt}) = a e^{-abt}$

and $a(1-b\,h(t)) = a(1 - b \cdot \frac{1}{b}(1-e^{-abt}))$

$= a(1 - 1 + e^{-abt}) = a e^{-abt}$.

So yes, $\frac{dh}{dt} = a(1-b\,h(t))$.

and $h(0) = \frac{1}{b}(1 - e^{-ab\cdot 0}) = \frac{1}{b}(1-1) = 0$.

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6.15 Suppose Ents grow to an average maximum height of 20m.
At 1 year of age their average height is 2m
(1) Find the average growth rate at birth.
(2) Find the average height at age 2.

$\frac{1}{b} = 20$ and $h(1) = 2$.

So $h(t) = \frac{1}{b}(1 - e^{-abt}) = 20(1 - e^{-\frac{a}{20}t})$

and $2 = h(1) = 20(1 - e^{-\frac{a}{20}})$. So $-\frac{1}{10} = e^{-a/20}$

So $2 = 20 - 20e^{-a/20}$ and $-18 = -20e^{-a/20}$.

So $\frac{9}{10} = e^{-a/20}$ and $-\frac{a}{20} = \log\left(\frac{9}{10}\right)$.

So $a = -20\log\left(\frac{9}{10}\right)$.

The average growth rate at birth is

$$a = -20\log\left(\frac{9}{10}\right) = 20\log\left(\frac{10}{9}\right)$$

the average height at age 2 is

$$h(2) = \frac{1}{b}(1 - e^{-ab\cdot 2}) = 20(1 - e^{-20\log\left(\frac{10}{9}\right)\frac{1}{20}\cdot 2})$$

$$= 20(1 - e^{-2\log(10/9)})$$

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Tutorial 10 Question 2

(d) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + c$, where c is a constant

since $\frac{d\arcsin(x)}{dx} = \frac{1}{\sqrt{1-x^2}}$.

(f) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \frac{1}{\sqrt{1-(ix)^2}}\,dx$. Let $u = ix$. Then $x = \frac{1}{i}u$ and $\frac{dx}{du} = \frac{1}{i}$.

So $\int \frac{1}{\sqrt{1-(ix)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\frac{dx}{du}\,du$

$= \frac{1}{i}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{i}\arcsin(u) + c$

$= \frac{1}{i}\arcsin(ix) + c$, where c is a constant.

Tutorial 10 Question 2

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(h) $\displaystyle\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{((x+i)(x-i))^2}\,dx$

$\displaystyle = \int \frac{1}{(x+i)^2(x-i)^2}\,dx = \int \frac{1}{2i}\,\frac{((x+i)-(x-i))}{(x+i)^2(x-i)^2}\,dx$

$\displaystyle = \frac{1}{2i}\int\left(\frac{1}{(x+i)(x-i)^2} - \frac{1}{(x+i)^2(x-i)}\right)dx$

$\displaystyle = \frac{1}{2i}\int\left(\frac{\frac{1}{2i}((x+i)-(x-i))}{(x+i)(x-i)^2} - \frac{\frac{1}{2i}((x+i)-(x-i))}{(x+i)^2(x-i)}\right)dx$

$\displaystyle = \frac{1}{(2i)^2}\int\left(\frac{1}{(x-i)^2} - \frac{1}{(x+i)(x-i)} - \frac{1}{(x+i)(x-i)} + \frac{1}{(x+i)^2}\right)dx$

$\displaystyle = \frac{-1}{4}\int\left((x-i)^{-2} - 2\cdot\frac{1}{(x^2+1)} + (x+i)^{-2}\right)dx$

$\displaystyle = \frac{-1}{4}\left(-(x-i)^{-1} - 2\arctan(x) - (x+i)^{-1}\right) + c$

$\displaystyle = \frac{1}{4}\left(\frac{1}{x-i} + \frac{1}{x+i} + 2\arctan(x)\right) + c$

$\displaystyle = \frac{1}{4}\left(\frac{x+i+x-i}{(x-i)(x+i)} + 2\arctan(x)\right) + c$

$\displaystyle = \frac{1}{4}\left(\frac{2x}{x^2+1} + 2\arctan(x)\right) + c$

$\displaystyle = \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan(x) + c$, where c is a constant.

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Tutorial 10 Question 2

(l) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx = -\int x\cdot\frac{-2x}{2\sqrt{1-x^2}}\,dx$

$$= -\int\left(x\cdot\frac{-2x}{2\sqrt{1-x^2}} + 1\cdot\sqrt{1-x^2} - \sqrt{1-x^2}\right)dx$$

$$= -\int\left(x\cdot\frac{-2x}{2\sqrt{1-x^2}} + 1\cdot\sqrt{1-x^2} - \frac{1-x^2}{\sqrt{1-x^2}}\right)dx$$

$$= -\int\left(x\cdot\frac{-2x}{2\sqrt{1-x^2}} + 1\cdot\sqrt{1-x} - \frac{1}{\sqrt{1-x^2}}\right)dx$$

$$-\int\frac{x^2}{\sqrt{1-x^2}}\,dx.$$

So $\displaystyle\int\frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{-1}{2}\int\left(x\cdot\frac{-2x}{2\sqrt{1-x^2}} + 1\cdot\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right)dx$

$$= \frac{-1}{2}\left(x\sqrt{1-x^2} - \arcsin(x)\right) + c,$$

where c is a constant.