Example 4.34b Find $\frac{dy^3}{dx}$.

The derivative $\frac{d}{dx}$ satisfies:

(1) If $c_1, c_2 \in \mathbb{C}$ then

$$\frac{d(c_1 f + c_2 g)}{dx} = c_1 \frac{df}{dx} + c_2 \frac{dg}{dx}.$$ (linearity)

(2) $$\frac{d(fg)}{dx} = f\frac{dg}{dx} + \frac{df}{dx} g$$ (product rule)

(3) If $c \in \mathbb{C}$ then $\frac{dc}{dx} = 0$ (Normalization)

and $\frac{dx}{dx} = 1$

Then

$$\frac{dy^3}{dx} = \frac{d(y \cdot y^2)}{dx} = y\frac{dy^2}{dx} + \frac{dy}{dx}y^2$$

$$= y\left(\frac{d(y\cdot y)}{dx}\right) + y^2\frac{dy}{dx} = y\left(y\frac{dy}{dx} + \frac{dy}{dx}y\right) + y^2\frac{dy}{dx}$$

$$= y^2\frac{dy}{dx} + y^2\frac{dy}{dx} + y^2\frac{dy}{dx} = 3y^2\frac{dy}{dx}$$ //

Theorem If $n \in \mathbb{Z}_{>0}$ then

$$\frac{dy^n}{dx} = ny^{n-1}\frac{dy}{dx} \quad \text{and} \quad \frac{dx^n}{dx} = nx^{n-1}.$$

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Example 4.34 Find $\frac{de^y}{dx}$.

$$\frac{de^y}{dx} = \frac{d}{dx}\left(1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\frac{1}{4!}y^4+\cdots\right)$$

$$= \frac{d1}{dx}+\frac{dy}{dx}+\frac{1}{2!}\frac{dy^2}{dx}+\frac{1}{3!}\frac{dy^3}{dx}+\frac{1}{4!}\frac{dy^4}{dx}+\cdots$$

$$= 0+\frac{dy}{dx}+\frac{1}{2\cdot 1}2y\frac{dy}{dx}+\frac{1}{3\cdot 2\cdot 1}3y^2\frac{dy}{dx}+\frac{1}{4\cdot 3\cdot 2\cdot 1}4y^3\frac{dy}{dx}+\cdots$$

$$= \frac{dy}{dx}\left(1+y+\frac{1}{2!}y^2+\frac{1}{3!}y^3+\frac{1}{4!}y^4+\cdots\right) = e^y\frac{dy}{dx}.$$

Example 4.34a Find $\frac{d\sin(y)}{dx}$.

$$\frac{d\sin(y)}{dx} = \frac{d}{dx}\left(\frac{1}{2i}\left(e^{iy}-e^{-iy}\right)\right)$$

$$= \frac{1}{2i}\left(\frac{de^{iy}}{dx}-\frac{de^{-iy}}{dx}\right)$$

$$= \frac{1}{2i}\left(e^{iy}\frac{d(iy)}{dx}-e^{-iy}\frac{d(-iy)}{dx}\right)$$

$$= \frac{1}{2i}\left(e^{iy}i\frac{dy}{dx}-e^{-iy}(-i)\frac{dy}{dx}\right)$$

$$= \frac{i}{2i}\left(e^{iy}+e^{-iy}\right)\frac{dy}{dx} = \cos(y)\frac{dy}{dx}.$$

Similarly $\frac{d\cos(y)}{dx} = -\sin(y)\frac{dy}{dx}$.

The chain rule

$$\frac{d(f\circ g)}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

i.e. $\frac{d(f(g(x)))}{dx} = \frac{df}{dg}\cdot\frac{dg}{dx}$. Examples are

$$\frac{dy^3}{dx} = 3y^2\frac{dy}{dx} \quad \text{and} \quad \frac{dy^3}{dy} = 3y^2$$

$$\frac{d\sin(y)}{dx} = \cos(y)\frac{dy}{dx} \quad \text{and} \quad \frac{d\sin(y)}{dy} = \cos(y).$$

$$\frac{de^y}{dx} = e^y\frac{dy}{dx} \quad \text{and} \quad \frac{de^y}{dy} = e^y.$$

Example 4.34

Find $\frac{d(\log(x))}{dx}$.

Let $y = \log(x)$. Then $e^y = e^{\log(x)} = x$.

Then $e^y = x$ gives $\frac{de^y}{dx} = \frac{dx}{dx}$.

So $e^y\frac{dy}{dx} = 1$. So $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

So $\frac{d\log(x)}{dx} = \frac{1}{x}$, and $\frac{d\log(y)}{dx} = \frac{1}{y}\frac{dy}{dx}$.

<u>4.26</u> Write down the first five derivatives of $f(x) = \sin(x)$

$f' = \frac{df}{dx} = \frac{d}{dx}(\sin(x)) = \cos(x)$,

$f'' = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(\cos(x)) = -\sin(x)$,

$f^{(3)} = f''' = \frac{d^3f}{dx^3} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) = \frac{d}{dx}(-\sin(x)) = -\cos(x)$

$f^{(4)} = f'''' = \frac{d^4f}{dx^4} = \frac{d}{dx}\left(\frac{d^3f}{dx^3}\right) = \frac{d}{dx}(-\cos(x)) = -(-\sin(x)) = \sin(x)$.

$f^{(5)} = f''''' = \frac{d^5f}{dx^5} = \frac{d}{dx}\left(\frac{d^4f}{dx^4}\right) = \frac{d}{dx}(\sin(x)) = \cos(x)$.

<u>4.27</u> Let $a, b, c, d \in \mathbb{R}$ and $f = ax^3 + bx^2 + cx + d$.
Find $f^{(n)}(x)$ for $n \le 6$.

$f^{(1)} = \frac{df}{dx} = \frac{d}{dx}(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c$

$f^{(2)} = \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(3ax^2 + 2bx + c) = 6ax + 2b$

$f^{(3)} = \frac{d^3f}{dx^2} = \frac{d}{dx}\left(\frac{d^2f}{dx^2}\right) = \frac{d}{dx}(6ax + 2b) = 6a$

$f^{(4)} = \frac{d^4f}{dx^4} = \frac{d}{dx}\left(\frac{d^3f}{dx^3}\right) = \frac{d}{dx}(6a) = 0$.

$f^{(5)} = \frac{d}{dx}\left(\frac{d^4f}{dx^4}\right) = \frac{d}{dx}(0) = 0$

$f^{(6)} = \frac{d}{dx}\left(\frac{d^5f}{dx^5}\right) = \frac{d}{dx}(0) = 0$.