

Combinatorics. An Introduction

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Abstract

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Contents

0 Preface	3
1 Introduction	4
1.1 What is combinatorics?	4
1.2 What is combinatorics not?	4
1.3 References	5
2 An introduction to counting	6
2.1 Counting functions	6
2.1.1 Exercises	11
2.2 The binomial theorem and multinomial coefficients	12
2.2.1 Exercises with Answers	13
2.3 Binomial coefficients	13
2.3.1 Exercises	17
2.4 Stirling numbers	17
2.4.1 Exercises	18
2.5 Lattice paths	19
2.5.1 Exercises.	20
2.6 Error correcting codes	20
2.6.1 Exercises	22
2.7 Problems	22
3 Basic combinatorial techniques	25
3.1 Inclusion-exclusion	25
3.1.1 Exercises with Answers	27
3.2 Generating functions	27
3.2.1 Exercises with Answers	30
3.3 Binomial inversion	30
3.4 Sieve formulas	33
3.4.1 Exercises with Answers	35
3.5 Finite integration	36
3.5.1 Exercises with answers	39

4 Pólya Theory	40
4.1 Burnside's Lemma	40
4.1.1 Exercises with Answers	44
4.2 The Cycle Index Polynomial	44
4.2.1 Exercises with Answers	46
4.3 Pólya's Inventory Theorem	46
4.3.1 Exercises with Answers	51
4.4 Problems 3	51
5 The Euler-Maclaurin summation formula	52
5.1 Bernoulli numbers and polynomials	52
5.1.1 Exercises with Answers	57
5.2 The Euler-Maclaurin Summation Formula	57
5.3 Problems 4	61
6 Finite operator calculus	62
6.1 Polynomial operators	62
6.1.1 Exercises with Answers	63
6.2 Differential operators	64
6.2.1 Exercises with Answers	65
6.3 Formulas of Maclaurin type	66
6.3.1 Exercises	67
6.4 Binomial sequences	67
6.4.1 Exercises	71
6.5 The first expansion theorem	72
6.5.1 Exercises with Answers	74
6.6 Sheffer sequences	74
7 Möbius Inversion	79
7.1 Partially ordered sets	79
7.2 The incidence algebra	80
7.2.1 Matrix representation of incidence functions	81
7.2.2 Exercises with Answers	82
7.3 Möbius inversion in partially ordered sets	83
7.3.1 Exercises with Answers	86
7.4 Applications	86
7.4.1 Exercises with Answers	90
8 Combinatorial species	91
8.1 Applications – An intuitive account	91
8.2 The species of Graphs	93
8.3 Definition of Species	95
8.4 Examples of species	97
8.5 Isomorphism of species	98
8.6 Sums and products of species	99
8.7 Some simple species	101
8.8 Substitution of species	103
8.9 Derivative of a species	107
8.10 Problems 7	109

9 Rota's method of linear functionals	111
9.1 Linear functionals	111
9.1.1 Exercises	115
9.2 Dobinski's formula	115
10 More finite operator calculus	117
10.1 The Algebra of Shift Invariant Operators	117
10.1.1 Exercises	118
10.2 The Pincherle Derivative	118
10.3 Applications	122
10.3.1 Exercises	127

0 Preface

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I owe a special debt to Don Taylor. Don is largely responsible for the introduction of modern combinatorial ideas to third year students at the University of Sydney. The courses he has developed have greatly influenced the choice and the presentation of many topics in this book.

Finally, I am very grateful to Wylie Breckenridge, Chris Bullivant, Humphrey Gastineau-Hills, Lesley Johnston, Adrian Nelson, Bill Unger, Bob Walters and Daniel Yee who have each read parts of the book, corrected many errors and made valuable comments.

Last but not least, I thank Bob Walters who created the diagrams in this book.

K.H. Wehrhahn

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1 Introduction

1.1 What is combinatorics?

Broadly, it is about combinations of objects, simple objects: like the natural numbers $1, 2, 3, 4, \dots$; or subsets of a set; or points and edges.

Because we are concerned with combining separate objects, combinatorics is often called *discrete mathematics*.

Combining simple objects is very basic: to mathematics, so most mathematics has gone through a combinatorial stage. Even π has combinatorial nature, as we see in the beautiful product of Wallis:

$$\frac{2 \cdot 2}{3}, \quad \frac{1 \cdot 2 \cdot 4 \cdot 4}{3 \cdot 3 \cdot 5}, \quad \frac{1 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7}, \quad \frac{1 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9}, \quad \dots \longrightarrow \pi$$

1.2 What is combinatorics not?

I think that can be summed up in a famous quote of Bertrand Russell, “Mathematics, rightly viewed, possesses . . . supreme beauty, cold and austere, like that of sculpture, without appeal to any part of our weaker nature.” Combinatorics is not like that. Combinatorics is more like a mountain meadow, filled with all sorts of interesting and beautiful flowers, which appeals to every part of our nature.

But there is order within the profusion that is combinatorics, and this book attempts to show some of the underlying connections and patterns, not only within combinatorics, but between the discrete and the continuous, between the finite and the infinitesimal. Four of the nine chapters (5, 6, 8 and 9) are concerned with pioneering work of Gian-Carlo Rota, whose series of papers “*On the Foundations of Combinatorial Theory*”, have done the most to bring order out of beautiful chaos. The first of these papers resulted in the award of the 1988 Steele prize to Rota, *for a paper of lasting and fundamental importance*.

This book is based on combinatorics courses given to third year students, at both ordinary and honours level, at the University of Sydney over the last ten years. The theme of the book is the theory of counting. Chapter 1 is concerned with elementary results, including the basic facts about binomial coefficients and Stirling numbers. In Chapter 2 we give a systematic treatment of some of the main techniques used in counting. Chapter 3 is devoted to Pólya theory, which uses group theory to count collections of objects possessing some symmetry.

The combinatorial identities which arise when counting a collection in different ways lead naturally to polynomial identities and in Chapters 4 and 5 and again in Chapters 8 and 9, we explore the interplay between counting and the calculus of polynomials.

In Chapter 6 we give an introduction to Rota’s theory of Möbius inversion in partially ordered sets, which brings together the diverse theories of inclusion-exclusion, and Möbius inversion in number theory.

Finally, in Chapter 7, we introduce the notion of Species of Structure due to André Joyal which has great promise and already many achievements, in providing a conceptual interpretation to the theory of generating functions.

This book draws on the mathematics background of third year students. For example, we assume a little matrix theory, some Fourier series in some of the problems of Chapter 4, elementary group theory in Chapter 3 and, in Chapter 7, some experience with functions between finite sets.

“*On the foundations of combinatorial theory I. Theory of Möbius functions.*” *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2 (1964) 340-368.

Based on the classic paper by Joyal, “*Une théorie combinatoire des séries formelles*”, *Advances in Math.* 42 (1981), 1-82.

The choice of topics reflects my interest in the subject and I have made some attempt, not enough, at telling a coherent story, at the expense of the omission of many interesting topics.

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