10 More finite operator calculus

In this chapter, we continue the developments of Chapters 4 and 5 with the aim of bringing together the powerful technique of generating functions and the finite operator methods. We will see how to find the binomial sequence of a delta operator. As applications, we will be able to derive integral quadratiure formulas, like Simpson's Rule. A final application, on the Laguerre polynomials, will point the way to the important area of polynomial sequences which arise as solutions of differential equations.

10.1 The Algebra of Shift Invariant Operators

In this section we establish the relationship between the algebra Σ of shift invariant operators and the algebra $\mathbb{R}[[t]]$ of formal power series over $\mathbb{R}$ in the variable t. The main result is the following *isomorphism theorem.*

Theorem 10.1. *The mapping*

$$\phi\colon f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k \longmapsto f(D) = \sum_{k\geq 0} \frac{a_k}{k!} D^k$$

is an algebra isomorpihsm between $\mathbb{R}[[t]]$ *and* Σ.

Proof. ϕ is clearly linear. By the First Expansion Theorem 5.3 of Chapter 5, ϕ is onto. To show that ϕ is one-to-one, suppose that $f(D) = 0$. Since

$$D^k x^n = \begin{cases} n_{(k)} x^{n-k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k, \end{cases}$$

we find that

$$f(D)x^n = a_n + \binom{n}{1} a_{n-1} x + \binom{n}{2} a_{n-2} x^2 + \cdots + a_0 x^n.$$

Hence $f(D)x^n = 0$ implies that $a_n = 0$ for each n. In other words, $f(t) = 0$ and so $\ker\ \phi = 0$ and so ϕ is a linear isomorphism. It remains to show that ϕ maps the product $f \cdot g(t) = f(t)g(t)$ of two power series into the composite $f(D)g(D)$ of the correspondng operators. Let

$$f(t) = \sum_{k\geq 0} \frac{a_k}{k!} t^k, \qquad g(t) = \sum_{k\geq 0} \frac{b_k}{k!} t^k.$$

Then

$$\begin{aligned}(f \cdot g)(t) &= \sum_{n\geq 0} \Big(\sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} \Big) t^n \\ &= \sum_{n\geq 0} \Big(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \Big) \frac{t^n}{n!}.\end{aligned}$$

We need to show that

$$(f \cdot g)(D) = f(D)g(D).$$

But by the First Expansion Theorem 5.3 of Chapter 5,

$$f(D)g(D) = \sum_{n\geq 0} \frac{c_n}{n!} D^n,$$

where

$$\begin{aligned} e_n = [f(D)g(D)]_{x=0} &= \Big[\Big(\sum_{k\ge 0}\frac{a_k}{k!}D^k\sum_{l\ge 0}\frac{b_l}{l!}D^l\Big)x^n\Big]_{x=0} \\ &= \Big[\Big(\sum_{k,l\ge 0}\frac{a_k}{k!}\frac{b_l}{l!}D^{k+l}\Big)x^n\Big]_{x=0} \\ &= \Big[\sum_{k,l\ge 0}\frac{a_k}{k!}\frac{b_l}{l!}n_{(k+l)}x^{n-k-l}\Big]_{x=0} \\ &= \sum_{k=0}^{n}\frac{a_kb_{n-k}}{k!(n-k)!}n! = \sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}. \end{aligned}$$

So $\sum_{n\ge 0}\frac{c_n}{n!}D^n = (f\cdot g)(D)$ and the proof is complete. □

Corollary 10.2. *The algebra Σ of shift invariant operators is commutative.*

Note. The power series $f(t) = 1$ is the multiplicative identity in $\mathbb{R}[[t]]$ and the formal power series

$$\sum_{k=0}^{\infty} a_k t^k$$

is invertible in $\mathbb{R}[[t]]$ if and only if $a_0 \neq 0$. Hence we have

Corollary 10.3. *A shift operator $T\in\Sigma$ is invertible if and only if $T1 \neq 0$. [i.e. $a_0\neq 0$.]*

10.1.1 Exercises

Let $P = \sum_{k\ge 0}\frac{a_k}{k!}D^k \in \Sigma$. Show that

(i) $\deg Pf(x) \le \deg f(x)$ for all $f\in\mathbb{R}[x]$.

(ii) P is invertible if and only if $a_0 \neq 0$.

(iii) P is invertible if and only if $\deg Pf(x) = \deg f(x)$ for all $f\in\mathbb{R}[x]$.

(iv) P is a delta operator if and only if $a_0 = 0$, $a_1 \neq 0$.

(v) P is a delta operator if and only if $P = DT$, where $T\in\Sigma$ and T is invertible.

10.2 The Pincherle Derivative

Problem. Given a delta operator Q how can we determine its normalized basis sequence?

To answer this we need the notion of *derivative of an operator* Q. But first we need to establish some notation.

Definition. Let T be shift invariant with $T = f(D)$ under the isomorphism of Theorem 6.1. Then we say that $f(t)$ is the *indicator* of T.

Let Q be a shift invariant operator. The *Pincherle derivative* of Q is the operator Q' whose indicator is the derivative of the indicator of Q. In other words, if $f(t)$ is the indicator of Q then $f'(t)$ is the indicator of Q'.

The following properties follow immediately from the definition.

Lemma 10.4. *For $P, Q \in \Sigma$, $\quad c \in \mathbb{R}$, the following are true:*

(i) $(P+Q)' = P' + Q'$,

(ii) $(cP)' = cP'$,

(iii) The derivative of a delta operator is invertible,

(iv) P' *is shift invariant.*

We need next a working formula for P'.

Lemma 10.5. *For any shift operator P, we have*

$$P' = Px - xP,$$

where x is the multiplication operator of §1.

Proof. By the First Expansion Theorem 5.3 of Chapter 5,

$$P = \sum_{k\geq 0} \frac{LPx^k}{k!} D^k,$$

and by definition of P', we have

$$P' = \sum_{k\geq 1} \frac{LPx^k}{(k-1)!} D^{k-1}.$$

But also $Px - xP$ is shift invariant if P is since

$$\begin{aligned}
E^a(Px - xP) &= E^aPx - E^axP \\
&= E^aPx - (x+a)E^aP \\
&= PE^ax - (x+a)PE^a \\
&= P(x+a)E^a - (x+a)PE^a \\
&= PxE^a - xPE^a + PaE^a - aPE^a \\
&= PxE^a - xPE^a \\
&= (Px - xP)E^a,
\end{aligned}$$

where we have used the facts that

$$E^ax = (x+a)E^a \qquad \text{and} \qquad aP = Pa \text{ for all } a \in \mathbb{R}.$$

Now using the Expansion theorem again we find

$$\begin{aligned}
Px - xP &= \sum_{k\geq 0} \frac{L(Px - xP)x^k}{k!} D^k \\
&= \sum_{k\geq 0} \frac{LPxx^k}{k!} D^k = \sum_{k\geq 0} \frac{LPx^{k+1}}{k!} D^k \\
&= P'.
\end{aligned}$$

□

Lemma 10.6. *Let $P, Q \in \Sigma$. Then* *(i)* $(PQ)' = PQ' + QP'$.

(ii) $(P^n)' = nP^{n-1}P'$ *for all integers* n, *provided* P *is invertible if* $n < 0$.

Proof. (i)

$$\begin{aligned}(PQ)' &= PQx - xPQ \\ &= PQx - PxQ + PxQ - xPQ \\ &= P(Q') + P'Q,\end{aligned}$$

so (i) follows by commutativity of shift invariant operators.
(ii) If $n > 0$ then (ii) follows by induction from (i). If $n = 0$, $P^0 = 1$ by definition and $I' = 0$. If $n < 0$ then

$$P^n = (P^{-1})^{|n|}$$

and so

$$(P^n)' = ((P^{-1})^{|n|})' = |n|(P^{-1})^{|n|-1}(P^{-1})' \quad \text{(from the } n > 0 \text{ case.)}$$

But $PP^{-1} = I$ so $(PP^{-1})' = 0$ and so

$$P(P^{-1})' + P'P^{-1} = 0.$$

Hence

$$(P^{-1})' = -P^{-1}P'.$$

Then for $n < 0$ we have

$$\begin{aligned}(P^n)' &= |n|(P^{-1})^{|n|-1}(P^{-1})^2 P' \\ &= nP^{n-1}P'\end{aligned}$$

as before. □

Observe that Lemma 2.3 yields two rules for differentiating formal power series:

$$\begin{aligned}\frac{d}{dt}(f(t)g(t)) &= f(t)g'(t) + f'(t)g(t), \\ \frac{d}{dt}f(t)^n &= nf'(t)f(t)^{n-1}.\end{aligned}$$

Example 1.

(1) $D' = 1$;

(2) $(E^a)' = E^a x - xE^a = (x+a)E^a = aE^a$;

(3) $\Delta' = (E - i)' = E$;

(4) $\nabla' = (1 - E^{-1})' = E^{-1}$.

We are now in a position to give a solution to the problem posed at the beginning of this section.

Theorem 10.7. *Let* $P = DT$ *be a delta oeprator. The normalized basis sequence of* P *is given by*

(i) $p_n(x) = P'T^{-n-1}x^n$, *for* $n \geq 0$;

(ii) $p_n(x) = xT^{-n}x^{n-1}$ *for* $n \geq 1$ *and* $p_0(x) = 1$; *[Steffensen's Formula.]*

(iii) $p_n(x) = x(P')^{-1}p_{n-1}(x)$ *for* $n \geq 1$ *and* $p_0(x) = 1$; *[Rodriques's Formula.]*

Proof. Let $p_n(x) = P'T^{-n-1}x^n$. Then

$$\begin{aligned} Pp_n(x) &= DTP'T^{-n-1}x^n \\ &= P'T^{-n}Dx^n \qquad \text{since } \Sigma \text{ is commutative,} \\ &= nP'T^{-n}x^{n-1} \\ &= np_{n+1}(x) \quad \text{for } n \geq 1. \end{aligned}$$

Also

$$p_0(x) = P'T^{-1}(1) = (DT'_T)T^{-1}(1) = 1,$$

and

$$P1 = 0.$$

Hence $\{p_n(x)\}$ is a basis sequence for P. To show it is normalized we prove that (i) and (ii) define the same sequence, [(ii) is clearly normalized],

$$\begin{aligned} xT^{-n}x^{n-1} &= (T^{-n}x - (T^{-n})')x^{n-1} \\ &= T^{-n}x^n + nT^{-n-1}T'x^{n-1} \\ &= T^{-n-1}(Tx^n + nT'x^{n-1}) \\ &= T^{-n-1}(T + T'D)x^n) \\ &= P'T^{-n-1}x^n). \end{aligned}$$

This shows that (i) and (ii) define the normalized basis sequence for P. To establish (iii), we have from (i)

$$x^n = (p')^{-1}T^{n+1}p_n(x),$$

and from (ii)

$$\begin{aligned} p_n(x) &= xT^{-n}x^{n-1} \\ &= xT^{-n}(P')^{-1}T^n p_{n-1}(x) \\ &= x(P')^{-1}p_{n-1}(x). \end{aligned}$$

□

Example 2.

(i) $P = DE^a$ [Abel's Operator].

By Theorem 2.4 (ii),

$$\begin{aligned} p_n(x) &= x(E^a)^{-n}x^{n-1} \qquad \text{for } n \geq 1,\ p_0(x) = 1, \\ &= x(x - na)^{n-1}. \end{aligned}$$

There are the *Abel Polynomials.*

(ii) $P = \Delta$ [The Forward Difference Operator]

Using Theorem 2.4 (iii) we have $\Delta' = E$ and so

$$\begin{aligned} p_n(x) &= x(\Delta')^{-1}p_{n-1}(x) = xE^{-1}p_{n-1}(x) \\ &= xp_{n-1}(x-1) = x(x-1)p_{n-2}(x-2) \\ &= \cdots = x(x-1)\cdots(x-n+1)p_0(x) \\ &= x_{(n)}. \end{aligned}$$

Corollary 10.8. *Let $P = DT$, $Q = DS$ be two delta operators with basis sequences $\{p_n(x)\}$ and $\{q_n(x)\}$ respectively. Then*

(i) $q_n(x) = x(Ts^{-1})^n x^{-1} p_n(x)$ *for* $n \geq 1$.

(ii) $q_n(x) = Q'(P')^{-1}S^{-n-1}T^{n+1}p_n(x)$ *for* $n \geq 1$.

Example 3.

$P = \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$ [The Central Difference Operator].

Note $\delta = E^{-\frac{1}{2}}(E-1) = E^{-\frac{1}{2}}(\Delta$ and $\Delta = DT$ for some invertible $T \in \Sigma$.

Hence

$$\delta = DTE^{-\frac{1}{2}}.$$

We know that $\{x_{(n)}\}$ is the basis for Δ so by Corollary 2.5 (i) the basis $\{q_n(x)\}$ for δ is given by

$$\begin{aligned} q_n(x) &= x(t(tE^{-\frac{1}{2}})^{-1})x^{-1}x_{(n)} \\ &= xE^{\frac{n}{2}}x^{-1}x_{(n)} = xE^{\frac{n}{2}}(x-1)_{(n-1)} \\ &= x(x+\frac{n}{2}-1)_{(n-1)} \quad \text{for } n \geq 1. \end{aligned}$$

Also $q_0(x) = 1$.

□

10.3 Applications

In this section we collect together two applications of the results so far in this chapter. The first shows how integral approximation formulae, like Simpson's rule fit into the finite operator scheme. The second takes a brief excursion into the important area of differential equations.

Newton-Cotes formulas

As you know, many of the important integrals of science cannot be expressed as *finite combinations* of the elementary functions:

$$x^n \quad e^x, \quad \ln x, \quad \sin x, \quad \sin^{-1} x, \quad \tanh x, \quad \operatorname{sech}^{-1} x, \quad \text{etc.}$$

For example:

$$\left.\begin{array}{l}\displaystyle\int_0^x \sin x^2\,dx \\[2ex] \displaystyle\int_0^x \cos x^2\,dx\end{array}\right\} \qquad \text{[Fresnel integrals important in optics]}$$

$$\left.\begin{array}{l}\displaystyle\int_0^x (1-a^2\sin^2 x)^{1/2}\,dx \\[2ex] \displaystyle\int_0^x \frac{1}{(1-a^2\sin^2 x)^{1/2}}\,dx\end{array}\right\} \qquad \text{[Elliptic integrals]}$$

These are often evaluated using suitable rules for numerical integration called *quadrature rules*. Examples include the *trapezoidal rule* and *Simpson's rule*. These are rules in which the value of a definite integral is approximated using the information obtained from a discrete set of data values.

Problem. Given a function $y = f(x)$, we want to calculate

$$A = \int_a^b f(x)\,dx,$$

(at least approximately).

The idea is to partition the interval $[a, b]$ into subintervals:

$$[a = x_0, x_1], \quad [x_1, x_2], \quad \ldots, \quad [x_{m-1}, x_m = b],$$

and then approximate the function $f(x)$ over each subinterval by a polynomial. Different polynomials yield different rules:

liines	—	trapezoidal rule
parabola	—	Simpson's 1/3 rule
cubic	—	Simpson's 3/8 rule
degree 6 polynomials	—	Weddle's rule

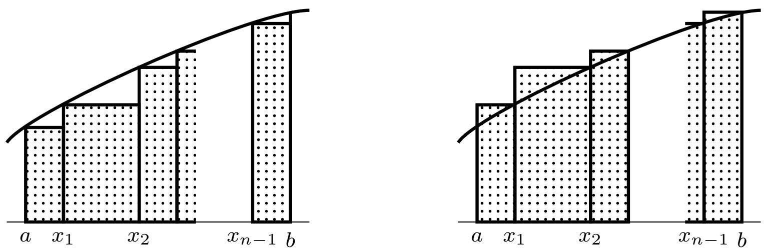

Figure 1.

Consider the first interval $[x_0, x_1]$ and suppose we want to approximate the function f by means of a polynomial of degree n. We then need to knowthe value of f at $n+1$ points $x_0, x_0+h, x_0+2h, \ldots, x_0+nh = x_1$, where $h = \dfrac{x_1 - x_0}{n}$.

We begin by making the following transformation:

$$t = \frac{x - x_0}{h}.$$

Then

$$\int_{x_0}^{x_1} f(x)\,dx = h\int_0^n f(x_0+ht)\,dt.$$

Let $g(t)$ be that unique polynomial of degree n for which

$$g(t) = f(x_0+ht).$$

for $t = 0, 1, 2, \ldots, n$.
Then we have the following approximation:

$$\int_{x_0}^{x_1} f(x)\,dx \approx h\int_0^n g(t)\,dt. \tag{1}$$

We can write the right hand side of (1) in terms of the *Bernoulli oeprator* J_n,

$$J_n g(x) = \int_x^{x+n} g(t)\,dt.$$

We have already studied $J_1 = J$, and we will relate the properties of J_n to those of J.

Properties of J_n

(1) J_n is shift invariant.

(2)

$$\begin{aligned} J_n g(x) &= \Big[\int_x^{x+1} + \int_{x+1}^{x+2} + \cdots + \int_{x+n-1}^{x+n}\Big] g(t)\,dt \\ &= \big(1 + E + \cdots + E^{n-1}\big) Jg(t). \end{aligned}$$

This suggests the compact form,

$$J_n g(x) = \frac{(1+\Delta^n) - I}{\Delta} Jg(t).$$

(3) By the First Expansion Theorem 5.3 of Chapter 5,

$$J = \sum_{k\geq 0} a_k \frac{\Delta^k}{k!},$$

where $a_k = \big[Jx_{(k)}\big]_{x=0} = \int_0^1 x_{(k)}\,dx$.

Hence we have the following expression for J_n:

$$\begin{aligned} J_n &= \frac{(1+\Delta^n) - I}{\Delta} J \\ &= \sum_{k=1}^{n} \binom{n}{k} \Delta^{k-1} \sum_{i\geq 0} a_i \frac{\Delta^i}{i!}. \end{aligned}$$

In particular,

$$a_1 = 1, \quad a_1 = \frac{1}{2}, \quad a_3 = -\frac{1}{6}, \quad a_3 = \frac{1}{4}, \quad a_4 = -\frac{19}{30}, \quad a_5 = \frac{9}{4}, \quad a_6 = -\frac{113}{84}, \quad \ldots.$$

Hence

$$\begin{aligned} J &= 1 + \tfrac{1}{2}\Delta - \tfrac{1}{12}\Delta^2 + \tfrac{1}{24}\Delta^3 - \tfrac{19}{720}\Delta^4 + \tfrac{9}{480}\Delta^5 - \cdots, \\ J_2 &= 2\big(1 + \Delta + \tfrac{1}{6}\Delta^2 - \tfrac{1}{180}\Delta^4 + \tfrac{1}{180}\Delta^5 + \cdots\big), \\ J_3 &= 3\big(1 + \tfrac{3}{2}\Delta + \tfrac{3}{4}\Delta^2 + \tfrac{1}{8}\Delta^3 - \tfrac{1}{80}\Delta^4 - \tfrac{1}{32}\Delta^5 \pm \cdots\big). \end{aligned} \tag{2}$$

Now returning to (1) we can write

$$\int_{x_0}^{x_1} f(x)\,dx \approx h\int_0^n g(t)\,dt = h\big[J_n g(x)\big]_{x=0}. \tag{3}$$

Example 4. (Simpson's 1/3-Rule.) Take $n = 2$ in (3). Then $g(t)$ is of degree 2 and so

$$\Delta^2 g(0) = \Delta^6 g(0) = \cdots = 0.$$

Hence it follows that

$$\begin{aligned} \big[J_2 g(x)\big]_{x=0} &= 2\big(g(0) + \Delta g(0) + \tfrac{1}{6}\Delta^2 g(0)\big) \\ &= \tfrac{1}{3}\big(f(x_0) + 4f(x_0+h) + f(x_1)\big), \end{aligned}$$

and we have Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(\begin{array}{l} f(x_0) + f(x_m) + 2(f(x_1) + \cdots + f(x_{m-1})) \\ \quad +4(f(x_0+h) + f(x_1+h) + \cdots + f(x_{m-1}+h)) \end{array}\right).$$

Laguerre polynomials

So far in this chapter we have studied certain polynomial sequences which arise in combinatorics. However, many of the most important polynomial sequences arise as solutions of differential equations. For example, the *Laguerre differential equation*,

$$xy'' + (1-x)y' + ny = 0 \tag{4}$$

has polynomial solutions $l'_n(x)$ where

$$\begin{aligned} &l'_0(x) = 1, \quad l'_1(x) = 1 - x, \quad l'_2(x) = 2 - 4x + x^2, \\ &l'_3(x) = 6 - 18x + 9x^2 - x^3, \quad \text{etc.} \end{aligned}$$

The sequence $\{l'_n(x)\}$ is clearly not of binomial type because it is not normalized. However it will turn out that it is a Sheffer sequence related to another famous delta operator Lg – the *Laguerre operator.*

Definition. The Laguerre operator is defined by

$$Lgp(x) = -\int_0^\infty e^{-t}\frac{d}{dx}p(x+t)\,dt. \tag{5}$$

We begin by applying the theory of the preceding sections to Lg and its basis sequence.

Properties of the Laguerre operator

Lemma 10.9. *Lg is a delta operator.*

Proof. We apply Lemma 4.3 of Chapter 5. It is easy to verify that Lg is shift invariant. Moreover,

$$Lgx = -\int_0^\infty e^{-t}\,dt = -1.$$

Hence Lg is a delta operator by Lemma 4.3. □

Lemma 10.10.

$$Lg = \frac{D}{D-1}.$$

Proof. Apply the First Expansion Theorem 5.3 of Chapter 5, to obtain

$$Lg = \sum_{n\ge 0} \frac{a_n}{n!} D^n,$$

where $a_n = [Lgx^n]_{x=0}$. For $n>0$ wee have

$$\begin{aligned}
Lgx^n|_{x=0} &= -\int_0^\infty e^{-t} n t^{n-1}\,dt \\
&= n\int_0^\infty t^{n-1}\,d(e^{-t}) \\
&= n\Big(t^{n-1}e^{-t}|_0^\infty - \int_0^\infty e^{-t}(n-1)t^{n-2}\,dt\Big) \\
&= -n(n-1)\int_0^\infty e^{-t}t^{n-2}\,dt \\
&= n(n-1)\int_0^\infty t^{n-2}\,d(e^{-t}) \\
&\ \ \vdots \\
&= -n!.
\end{aligned}$$

Also $Lg1 = 0$, so $a_0 = 1$. Hence

$$\begin{aligned}
Lg &= -\big(D = D^2 + D^3 + \cdots\big) \\
&= -\frac{D}{1-D} \\
&= \frac{D}{D-1}.
\end{aligned}$$

□

Lemma 10.11. *The basis sequence of Lg is $\{l_n(x)\}$ where*

$$l_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}\binom{n-1}{k-1} x^k.$$

Proof. Apply Stephenson's formula of Theorem 2.4 (ii) with

$$Lg = DT \qquad \text{where} \qquad T = \frac{1}{D-1}.$$

For $n \geq 1$ we get,

$$\begin{aligned} I_n(x) &= x\Big(\frac{1}{D-1}\Big)^{-n} x^{n-1} \\ &= x(D-1)^n x^{n-1} \\ &= x\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} D^k x^{n-1} \\ &= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \binom{n-1}{k-1} x^k. \end{aligned} \tag{6}$$

Clearly, $l_0(x) = 1$ and so the proof is complete. □

Note. From Lemma 3.3, it follows that the first few *basis Laguerre polynomials* are

$$\begin{aligned} l_0(x) &= 1, \\ l_1(x) &= -x, \\ l_2(x) &= -2x + x^2, \\ l_3(x) &= -6x + 6x^2 - x^3, \\ l_4(x) &= -24x + 36x^2 - 12x^3 + x^4. \end{aligned}$$

Note. We can write

$$D - I = e^x D e^{-x},$$

since for $f(x) \in \mathbb{R}[x]$ we have

$$\begin{aligned} e^x D e^{-x} f(x) &= e^x D\big(e^{-x} f(x)\big) \\ &= e^x\big(-e^{-x} f(x) + e^{-x} Df(x)\big) \\ &= Df(x) - f(x) \\ &= (D-1)f(x). \end{aligned}$$

Hence from (6) we have

$$\begin{aligned} l_n(x) &= x(e^x D e^{-x})^n x^{n-1} \\ &= xe^x D^n e^{-x} x^{n-1}. \end{aligned} \tag{7}$$

Equation (7) is the *classical Rodrigues formula* for the Laguerre polynomials.

10.3.1 Exercises

1. Take $n = 3$ in (3) to establish Simpson's $\frac{3}{8}$-rule.

$$\int_a^b f(x)\,dx \approx \frac{3h}{8} \begin{pmatrix} f(x_0) + f(x_m) \\ +3\big(f(x_0+h) + +f(x_1+h) + \cdots + f(x_{m-1}+h)\big) \\ +3\big(f(x_0+2h) + f(x_1+2h) + \cdots + f(x_{m-1}+2h)\big) \end{pmatrix}.$$

[The interval $[a, b]$ is partitioned into m equal subintervals and f is approximated by a cubic over each subinterval.]