9 Rota's method of linear functionals

In this seciton we will see how to translate relations between polynomials into relations between numbers. We use an idea developed by Gian-Carlo Rota.[1] For example, we will be able to prove Dobinski's formula,

$$\frac{1}{e}\Big(\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\frac{4^n}{4!}+\cdots\Big)=B_n,$$

where B_n $(n \geq 1)$ is the nth Bell number.

9.1 Linear functionals

First recall that any linear functional on $\mathbb{R}[x]$ (i.e. linear transformations of the form $f\colon \mathbb{R}[x] \to \mathbb{R}$) is uniquely determined by the values it takes on a basis. That is, for a polynomial sequence $\{p_n(x)\}$ and a sequence of numbers $\{a_n\}$ there is the unique linear functional L defined by

$$Lp_n(x) = a_n.$$

Example 1. Consider the sequence of standard polynomials $\{x^n\}$ and define L by

$$Lx^n = 1, \qquad \text{for } n = 0, 1, 2, \ldots.$$

(That is $\{a_n\} = \{1, 1, 1, \ldots\}$.) We extend L to the whole vector space of polynomials by linearity. Thus L maps each polynomial to the sum of its coefficients. For example,

$$L\sum_{k=0}^{n} c_n x^n = \sum_{k=0}^{n} c_n L x^n = c_0 + c_1 + \cdots + c_n.$$

Remark. We will usually take $\{a_n\}$ to be a very simple sequence, like $\{0, \ldots, 0, 1, 0, \ldots\}$ or $\{1, 1, 1, \ldots\}$. In a way, the linear functionals allow us to 'evaluate' a polynomial identity at a sequence of numbers. They can be used as an exploratory tool to find non-obvious relationships between sequences of numbers. To illustrate the method let's look at some examples.

Example 2. Consider the sequence $\{(x-1)^n\}$ (the standard polynomials shifted one unit to the right). Define L_k, for $k = 0, 1, 2, \ldots$, by

$$\begin{aligned} L_k(x-1)^n &= \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases} \\ &= \delta_{n,k} \quad \text{(for short).} \end{aligned} \tag{2}$$

(Extend L_k to the whole vector space of polynomials by linearity.)

Let us apply our linear functionals to the following identities,

$$x^n = (x-1+1)^n = \sum_{i=0}^{n} \binom{n}{i} (x-1)^i, \tag{3}$$

$$(x-1)^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x^i \tag{4}$$

[1] See Rota's *The Number of Partitions of a set*, American Mathematical Monthly 71 (1964), 498-504.

Applying L_k to (3) yields

$$L_k x^n = \sum_{i=0}^{n} \binom{n}{i} L_k (x-1)^i = \sum_{i=0}^{n} \binom{n}{i} \delta_{i,k}$$
$$= \binom{n}{k}. \tag{5}$$

Applying L_k to (4) yields,

$$L_k(x-1)^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} L_k x^i = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i}\binom{i}{k} \qquad \text{(by (5))}.$$

Here we obtain the identity of Problem 1 (ii), Exercises 1.3.

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i}\binom{i}{k} = \begin{cases} 1, & \text{if } n=k, \\ 0 & \text{otherwise}, \end{cases}$$
$$= \delta_{n,k}. \tag{6}$$

Example 3. Applying L_k of Example 2 to the recurrence equation

$$(x-1)^{n+1} = x(x-1)^n - (x-1)^n$$

yields

$$\begin{aligned} L_k(x-1)^{n+1} &= \delta_{n+1,k} = \delta_{n,k-1} = L_{k-1}(x-1)^n \\ &= L_k x(x-1)^n - L_k(x-1)^n \\ &= (L_k x - L_k)(x-1)^n, \end{aligned} \tag{7}$$

where x is the linear transformation of $\mathbb{R}[x]$ defined by

$$x f(x) = x f(x), \qquad \text{for all } f(x) \in \mathbb{R}[x].$$

By (7),

$$L_{k-1} = L_k x - L_k, \tag{8}$$

and so in particular, applying (8) to x^n, we find

$$L_{k-1}x^n = L_k x x^n - L_k x^n.$$

This yields the important recurrence relation for the binomial coefficients:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \tag{9}$$

Example 4. Let us try the same method on the Stirling numbers of the second kind which are defined as connection coefficients by,

$$x^n = \sum_{k=0}^{n} S(n,k) x_{(k)}. \tag{10}$$

In this case we define the linear functional L_k by

$$L_k x_{(n)} = \delta_{n,k}. \tag{11}$$

Applying L_k to (10) we find

$$L_k x^n = \sum_{i=0}^{n} S(n,i)\delta_{i,k} = S(n,k). \tag{12}$$

Applying L_k to the identity,

$$x_{(n)} = \sum_{k=0}^{n} s(n,k)x^k, \tag{13}$$

gives

$$L_k x_{(n)} = \delta_{n,k} = \sum_{i=0}^{n} s(n,i)S(i,k),$$

which yields again the identity

$$\sum_{k=0}^{n} s(l,k)S(k,j) = \delta_{l,j}. \tag{14}$$

Example 5. To obtain a recurrence relation for the $S(n,k)$, apply L_k to the polynomial recurrence

$$x_{(n+1)} = x_{(n)}(x-n). \tag{15}$$

Then

$$\begin{aligned} L_k x_{(n+1)} = L_{k-1}x_{(n)} &= L_k x x_{(n)} - nL_k x_{(n)} && (9.1)\\ &= L_k x x_{(n)} - kL_k x_{(n)}, && (9.2) \end{aligned}$$

which extends to all polynomials to give the identity

$$L_{k-1} = L_k x - kL_k. \tag{16}$$

Hence applying (16) to x^n yields

$$L_{k-1}(x^n) = L_k x x^n - kL_k x^n$$

or

$$S(n,k-1) = S(n+1,k)kS(n,k).$$

In other words, we have the recurrence for the Stirling numbers of the second kind,

$$S(n+1,k) = S(n,k-1) + kS(n,k) \tag{17}$$

Example 6. On the other hand, we can apply L_k to the polynomial recurrence

$$x_{(n+1)} = x(x-1)_{(n)}$$

to get

$$L_k x_{(n+1)} = L_{k-1}x_{(n)} = L_k x(x-1)_{(n)} \tag{18}$$

Let E^{-1} be the linear transformation defined by

$$E^{-1}p(x) = p(x-1).$$

[This is an example of a shift operator, which we first encountered in Chapter 4.]
Then (18) shows that

$$L_{k-1} = L_k x E^{-1}. \tag{19}$$

Evaluating both sides of (19) on the polynomial $(x+1)^n$ yields,

$$L_{k-1}(x+1)^n = L_k x E^{-1}(x+1)^n = L_k x^{n+1} = S(n+1,k).$$

In other words,

$$S(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} S(j,k-1), \tag{20}$$

(see Problem 2, Exercises 1.5).

One last example to illustrate the range of applicability of the linear functional method.

Example 7. Recall that the Bell number B_n denotes the *total* number of partitions of an n-set. Hence we can write

$$B_n = \sum_{k=0}^{n} S(n,k).$$

Define the linear functional L by

$$Lx_{(k)} = 1, \qquad \text{for } k = 0, 1, 2, \ldots,$$

then,

$$Lx^n = L\Big(\sum_{k=0}^{n} S(n,k)x_{(k)}\Big) = \sum_{k=0}^{n} S(n,k) = B_n.$$

Applying L to the polynomial recurrence

$$x_{(n+1)} = x(x-1)_{(n)}$$

yields the linear functional identity:

$$L = LxE^{-1}.$$

Hence

$$L(x+1)^n = Lx^{n+1}$$

or

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \tag{21}$$

which is a recurrence relation for the Bell numbers (see Problem 4, Exercises 1.5).

9.1.1 Exercises

1. The Strling numbers of the first kind, $s(n,k)$, are defined by

$$x_{(n)} = \sum_{k=0}^{n} s(n,k)x^k.$$

Use the methods of this section to establish the following recurrence relations analogous to (17) and (20):

(i) $s(n+1,k) = s(n,k-1) - ns(n,k)$.

(ii) $s(n+1,k) = \sum_{j=0}^{n}(-1)^j n_{(j)} s(n-j,k-1)$.

9.2 Dobinski's formul;a

In this section we use the linear functional technique to prove Dobinski's formula for the bell numbers B_n:

$$B_n = \frac{1}{e}\sum_{k\geq 0}\frac{k^n}{k!} \tag{22}$$

First, define $L\colon \mathbb{R}[x] \to \mathbb{R}$ by

$$Lx_{(k)} = 1 \qquad \text{for } k = 0,1,2,\ldots.$$

If we apply L to (10), the defining relation for $S(n,k)$, we get,

$$Lx^n = \sum_{k=0}^{n} S(n,k) = B_n \qquad \text{(the } n\text{th Bell number).} \tag{23}$$

On the other hand, recall,

$$e = \sum_{k\geq 0}\frac{1}{k!} = \sum_{k\geq 0}\frac{k_{(n)}}{k!} \qquad \text{for } n \geq 0.$$

Hence

$$1 = \frac{1}{e}\sum_{k\geq 0}\frac{k_{(n)}}{k!}$$

or

$$Lx_{(n)} = \frac{1}{e}\sum_{k\geq 0}\frac{k_{(n)}}{k!}.$$

I want to evaluate $\frac{1}{e}\sum_{k\geq 0}\frac{p(k)}{k!}$ for any polynomial $p(x) = a_0 + a_1x_{(1)} + \cdots + a_nx_{(n)}$.

But,

$$\frac{1}{e}\sum_{k\geq 0}\frac{p(k)}{k!} = \frac{1}{e}\sum_{k\geq 0}\frac{1}{k!}\sum_{l=0}^{n} a_l k_{(l)}$$
$$= \frac{1}{e}\begin{pmatrix} \frac{1}{0!}(a_0 0_{(0)} + a_1 0_{(1)} + \cdots + a_n 0_{(n)}) \\ +\frac{1}{1!}(a_0 1_{(0)} + a_1 1_{(1)} + \cdots + a_n 1_{(n)}) \\ \vdots \\ +\frac{1}{k!}(a_0 k_{(0)} + a_1 k_{(1)} + \cdots + a_n k_{(n)}) \\ \vdots \end{pmatrix}.$$

This infinite series is absolutely convergent, so we can intechange summations to get,

$$\frac{1}{e}\sum_{k\geq 0}\frac{p(k)}{k!} = \frac{1}{e}\sum_{k\geq 0}\frac{1}{k!}\sum_{l=0}^{n} a_l k_{(l)} = \frac{1}{e}\sum_{l\geq 0} a_l \sum_{l=0}^{n}\frac{k_{(l)}}{k!} = \sum_{l=0}^{n} a_l$$
$$= Lp(x).$$

Letting $p(x) = x^n$ and using (23), we get Dobinski's formula.