7 Möbius Inversion

Our aim in this chapter is to look at the valuable idea of inclusion-exclusion which we studied in §1 and §4 of Chapter 2. We will show how this notion may be generalized to the setting of *partially ordered sets*. Our account is basded on the first of the *Foundations of Combinatorics* papers by Gian-Carlo Rota, "*On the foundations of combiantorial theory I. Theory of Möbius functions*", Z. Wahrscheinlichkeitstheorie 2 (1964) 340-368.

7.1 Partially ordered sets

Definition. Let P be a set (not necessarily finite). A *partial order* on P is a binary relation $\le$ which is

(i) reflexive, i.e.,
$$x \le x, \qquad \text{for all } x \in P;$$

(ii) transitive, i.e.,
$$x \le y \quad \text{and} \quad y \le z \quad \Rightarrow \quad x \le z, \qquad \text{for all } x, y, z \in P;$$

(iii) antisymmetric, i.e.,
$$x \le y \quad \text{and} \quad y \le x \Rightarrow x = y, \qquad \text{for all } x, y \in P.$$

We say that P is a *partially ordered set*, or *poset*, if it has a partial order.

We say that x and y are *comparable* if $x \le y$ or $y \le x$.

Definition. The set $[x, y] = \{z \mid x \le z \le y\}$ is called the *segment*, or *interval*, from x to y.

Notation. Two elements in P need not be comparable, but if every pair of elements of P is comparable, we say that $\le$ is a *total order*, or a *linear order*. A totally ordered set is called a *chain*. The poset P is called *locally finite* if every interval of P is a finite set.

We say that y *covers* x if the segment $[x, y]$ contains only x and y, i.e. $[x, y] = \{x, y\}$.

Convention. $x < y$ means $x \le y$ but $x \ne y$; $x \ge y$ means $y \le x$.

Example 1. The two examples of greatest importance to us are:

(1) The set of subsets of a set, ordered by inclusion. That is, let X be a set and P the set of subsets of X. Then P is a poset with the inclusion realation $\subseteq$ as the partial order.

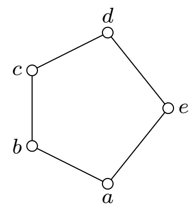
(2) The set of natural numbers ordered by divisibility. That is, let $\mathbb{N}$ be the set of natural numbers $\{1, 2, 3, \dots, \}$. Let $n \le m$ if $n|m$. Then $\mathbb{N}$ is a poset with $\le$ as the partial order.

Hasse diagrams

Using an idea due to the German mathematician Heinrich Hasse (1898-), we can display small posets and segments of posets by drawing a line from x up to y, whenever y *covers* x.

Rota won the Steele prize "for a paper of lasting and fundamental importance" for this paper. See Stanley, Richard P., "*Enumerative combinatorics Volume I*", Wadsworth & Brooks/Cole Mathematics Series 1986, Chapter 3, for a thorough account.

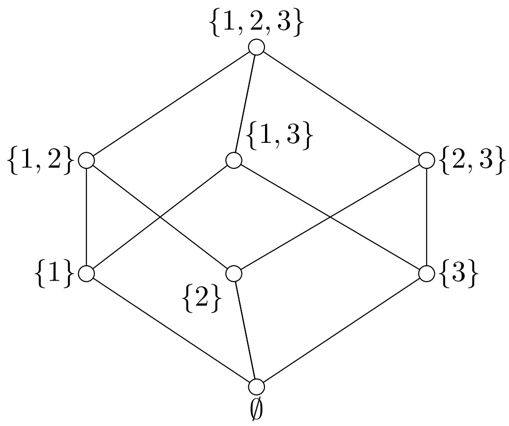
Example 2. Consider $P = \{a, b, c, d, e\}$, with partial order, $a \le b \le c \le d$, $a \le c \le d$. Its Hasse diagram is,



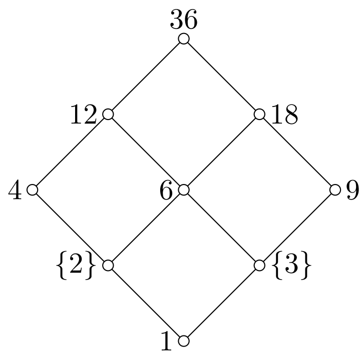
Example 3. Let $X = \{1, 2, 3\}$, P the poset of subsets of X, ordered by inclusion. Then

$$P = \big\{\emptyset, \{1\}, \{2\}, \{3, \}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

It's diagram is



Example 4. Let $\mathbb{N}$ be the natural numbers ordered by divisibility. Then the segment $[1, 30]$ can be displayed by:



7.2 The incidence algebra

Let P be a locally finite poset and $\mathcal{A}(P)$ the set of all functions

$$f : P \times P \to \mathbb{R},$$

which satisfy the condition

$$f(x, y) = 0, \qquad \text{whenever } x \not\le y.$$

[That is, $f(x, y) \neq 0$ only when $[x, y]$ is a *non-empty* interval.]

The element of $\mathcal{A}(P)$ can be multiplied by real numbers,

$$(\alpha f)\colon (x, y) \longmapsto \alpha(f(x, y)) \ ;$$

added together,

$$(f + g)\colon (x, y) \longmapsto f(x, y) + g(x, y);$$

and multiplied (by the convolution product),

$$(f * g)\colon (, xy) \longmapsto \sum_{z \in [x,y]} f(x, z)g(z, y).$$

Comment. With respect to these three operations, $\mathcal{A}(P)$ is an associative algebra over $\mathbb{R}$, called the *incidence algebra* of the poset P. The elements of $\mathcal{A}(P)$ are called *incidence functions*.

7.2.1 Matrix representation of incidence functions

If P is a finite poset we can identify the elements of $\mathcal{A}(P)$ with upper triangular matrices. To do this we need first a linear ordering of the elements of P, say $x_1, x_2, \ldots, x_n$. The only restriction on this ordering will be that it preserves the partial order, i.e., $x_i \le x_j \Rightarrow i \le j$.

Once the linear ordering has been decided, we associate $f \in \mathcal{A}(P)$ (with respect to the chosen ordering), the $n \times n$ matrix $(f(x_i, x_j))$, whose (i, j)th entry is $f(x_i, x_j)$.

Example 5. If P is the poset of Example 1, we can order the elements by a, b, c, e, d. Then the matrices associated with the incidence functions have the form:

$$\begin{array}{c} \\ a \\ b \\ c \\ e \\ d \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & e & d \end{array} \\ \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & 0 & * \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \end{array}$$

Note. Addition and multiplication of incidence functions now corresponds to matrix addition and multiplication. For example,

$$\begin{aligned}(f * g)(x_i, x_j) &= \sum_{z \in \{x_i, x_j\}} f(x_i, z)g(z, x_j) \\ &= \sum_{z \in P} f(x_i, z)g(z, x_j).\end{aligned}$$

[We just get a zero contribution for $z \notin [x_i, x_j]$.]

The identity element of $\mathcal{A}(P)$ is the function δ, defined by,

$$\delta(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

The algebra $\mathcal{A}(P)$ is isomorphic to the algebra $\mathcal{M} = (m_{i,j})$, of upper triangular matrices having $m_{i,j} = 0$ if $x_i \not\le x_j$.

If P is finite, δ is identified with the identity matrix.

Two important functions

(1) The *zeta function* ζ is defined by

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The *Möbius functions* μ is the inverse of the zeta function ζ. That is, μ satisfies the conditions,

$$\mu * \zeta = \zeta * \mu = \delta. \tag{1}$$

We construct the inverse of ζ recursively. We wnat μ such that for each $x, y \in P$,

$$\begin{aligned} \delta = \mu * \zeta(x,y) &= \sum_{z\in[x,y]} \mu(x,z)\zeta(z,y) \\ &= \sum_{z\in[x,y]} \mu(x,z). \end{aligned}$$

In particular, we need,

$$\mu(x,x) = \delta(x,x) = 1, \qquad \text{for all } x \in P. \tag{2}$$

If $x \neq y$ we need,

$$\sum_{z\in[x,y]} \mu(x,z) = 0,$$

which will be satisfied if we define μ recursively by,

$$\mu(z,x) = \sum_{x\leq z<y} \mu(x,z). \tag{3}$$

Example 6. Consider the poset of Examples 1 and 5. The matrix of zeta is

$$Z = \begin{array}{c} \\ a \\ b \\ c \\ e \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & e & d \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

The Möbius function μ, is given by $\mu(x,x) = 1$ for all $x \in P$, and

$$\begin{aligned} &\mu(a,b) = -\mu(a,a) = -1, \\ &\mu(a,c) = -\mu(a,a) - \mu(a,b) = 0, \\ &\mu(a,e) - \mu(a,a) - \mu(a,b) - \mu(a,c) - \mu(a,e) = 1, \\ &\mu(b,c) = -\mu(b,b) = -1, \\ &\mu(b,d)0, \quad \mu(c,d) = -1, \quad \mu(e,d) = -1. \end{aligned}$$

Hence the matrix of μ is:

$$M = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note. If P is finite it is clear that any f has an inverse in $\mathcal{A}(P)$ if and only if its matrix has only non-zero elements along the diagonal (i.e. if its determinant is non-zero).

7.2.2 Exercises with Answers

1. Let P be a partially ordered set.

(i) So that $\zeta^2(x, y) = |[x, y|$.

(ii) If $\eta = \zeta - \delta$, show that $\eta^k(x, y)$ is the number of chains of length k from x to y.

7.3 Möbius inversion in partially ordered sets

In this section we give the most general of our inversion formulas. First some definitions.

Definition.. An *order ideal* of P is a subset I of P such that whenever $x \in I$ and $y \leq x$ then $y \in I$. The *principal order ideal* generated by $x \in P$ is $\{y \mid y \leq x\}$.

A *dual order ideal* (also called a *filter*) of P is a subset $\tilde{I}$ of P such that whenever $x \in \tilde{I}$ and $x \leq y$ then $y \in \tilde{I}$. The *principal dual order ideal* generated by $x \in P$ is the set $\{y \mid x \leq y\}$.

Theorem 7.1.

(1) Let P be a poset in which each principal order ideal is finite. Then

$$t(x) = \sum_{y \leq x} e(y)$$

if and only if

$$e(x) = \sum_{y \leq x} t(y)\mu(y, x). \tag{4}$$

(2) Let P be a poset in which each principal dual order ideal is finite. Then

$$s(x) = \sum_{x \leq y} e(y)$$

if and only if

$$e(x) = \sum_{x \leq y} \mu(x, y)s(y). \tag{5}$$

Proof. (1) ($\Rightarrow$) Suppose $t(x) = \sum_{y \leq x} e(y)$ for all $x \in P$. Then

$$\begin{aligned} \sum_{y \leq x} t(y)\mu(y, x) &= \sum_{y \leq x} \Big(\sum_{x \leq y} e(z)\Big)\mu(y, x) \\ &= \sum_{z \leq x} e(z) \sum_{z \leq x \leq y} \mu(y, x). \end{aligned}$$

By (2) and (3), the second sum is $\begin{cases} 1, & \text{if } x = z; \\ 0, & \text{otherwise.} \end{cases}$.

Hence $\sum_{y \le x} t(y)\mu(y,x) = e(x)$. The symmetry of this argument also proves the converse.

The proof of (2) is similar. □

Note. If P is fintie and E, T are the *row* vectors associated with e and t, respectively then (1) says that,

$$T = EZ \quad \Longleftrightarrow \quad E = TM.$$

If E', S are the column vectors associated with e and s respectively, then (2) says,

$$S = ZE' \quad \Longleftrightarrow \quad E' = MS.$$

Example 7. Let P be the poset of Example 6 and suppose that $s\colon P \to \mathbb{R}$ is given by,

$x \in P$	a	b	c	e	d
$s(x)$	5	-1	0	4	1

Suppose that $s(x0 = \sum_{x \le y} e(y)$ for all $x \in P$. To find e, we observe that

$$S = ZE = \begin{pmatrix} 5 \\ -1 \\ 0 \\ 4 \\ 1 \end{pmatrix}$$

Hence $E = MS = \begin{pmatrix} 7 \\ -5 \\ -1 \\ 3 \\ 1 \end{pmatrix}$. Alternatively, e is given by the table,

$x \in P$	a	b	c	e	d
$e(x)$	7	-5	-1	3	1

Similarly, if $T = (5, -1, 0, 4, 1) = EZ$, then $E = TM = (5, -6, -5, 5, 2)$.

Example 8. Let P be the natural numbers $\mathbb{N}$ and let $\le$ have its usual meaning, e.g. $1 \le 2 \le 3 \le 4 \le \cdots$. Then P is a totally ordered set with respect to $\le$, and by (2), $\mu(i,i) = 1$ for $i \in \mathbb{N}$. By (3), $\mu(i, i+1) = -\mu(i,i) = -1$. It follows that,

$$\mu(i,j) = \begin{cases} 1, & \text{if } i = j; \\ -1, & \text{if } j = i+1; \\ 0, & \text{otherwise.} \end{cases}$$

Möbius inversion now takes the form,

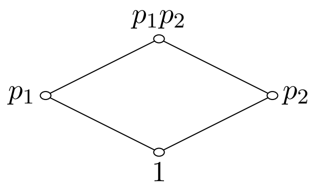
$$t(n) = \sum_{i=1}^{n} e(i), \quad \text{for all } n \ge 1 \qquad \Longleftrightarrow \qquad e(n) = \sum_{i=1}^{n} t(i)\mu(i,n) = t(n) - t(n-1) = \Delta t(n-1).$$

Hence, Möbius inversion in this case is just the Fundamental theorem of finite integration which we found in Chapter 2 §5.

Example 9. Let P be the natural numbers $\mathbb{N}$, but this time ordered by divisibility, i.e. $m \le n$ if and only if $m|n$. Let us calculate $\mu(m,n)$.
First suppose $n = p$ (a prime), then

$$\mu(1,p) = -\mu(1,1) = -1, \qquad \text{by (2)}.$$

If $n = p_1p_2$ (a product of two primes), the Hasse diagram of $[1,n]$ is



Now $\mu(1,p_1p_2) = -\mu(1,p_1) - \mu(1,p_2) - \mu(1,1) = 1+1-1 = 1$ (independently of the particular primes).
In general, if n is the product of distinct primes, say $n = p_1p_2\cdots p_k$, then $\mu(1,n) = (-1)^k$. To see this, just observe that,

$$\begin{aligned}
\mu(1,p_1p_2\cdots p_k) &= -\Big(1 + \binom{k}{1}\mu(1,p_1) + \binom{k}{2}\mu(1,p_1p_2) + \cdots + \binom{k}{k-1}\mu(1,p_1p_2\cdots p_{k-1})\Big) \\
&= -\Big(1 + \binom{k}{1}(-1)^1 + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k-1}(-1)^{k-1}\Big) \qquad \text{(by induction)} \\
&= -\Big((1-1)^k - \binom{k}{k}(-1)^k\Big) = (-1)^k.
\end{aligned}$$

Next suppose that n has a factor which is the square of a prime. The first case is $n = 4$, and clearly $\mu(1,4) = 0$. In fact if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime decomposition of n and $\alpha_i > 1$ for some i, then we will prove by induction that $\mu(1,n) = 0$.
Suppose $\mu(1,m) = 0$ if m is less than n and contains the square of a prime. Then,

$$\begin{aligned}
\mu(1,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}) &= -\Big(\mu(1,p_1p_2\cdots p_k) + \binom{k}{1}\mu(1,p_1p_2\cdots p_k) + \cdots + \binom{k}{k}\mu(1,1)\Big) \qquad \text{(by induction)} \\
&= -\Big((-1)^k + \binom{k}{1}(-1)^{k-1} + \cdots + \binom{k}{k}(-1)^0\Big) \\
&= -(1-1)^k = 0.
\end{aligned}$$

The general case of $\mu(m,n)$ where $n = mp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ now follows immediately since the segment $[m,n]$ is identical to $[1,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}]$, where mx of $[m,n]$ corresponds to x of $[1,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}]$.
Hence for $n = mp_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, we have,

$$\mu(m,n) = \mu(1,p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}) = \begin{cases} 1, & \text{if } m = n, \\ (-1)^k, & \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Möbius inversion now takes the following form.

$$t(n) = \sum_{d|n} e(d) \quad \Longleftrightarrow \quad e(n) = \sum_{d|n} t(d)\mu(d,n) = \sum_{d|n} \bar{\mu}(\frac{n}{d})t(d), \tag{7}$$

where $\bar{\mu}$ is the usual Möbius function of number theory, $\bar{\mu}(n) = \mu(1,n)$.

Example 10. Let X be a finite set and P the set of all subsets of X, ordered by inclusion. That is, if $I, J \in P$ the $I \le J$ means $I \subseteq J$, We will show that the Möbius function for P is,

$$\mu(I,J) = \begin{cases} (-1)^{|J|-|I|}, & \text{if } I \subseteq J, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The proof is by induction on $|J| - |I|$.
$\mu(I,I) = 1$, so (8) holds if $|J| - |I| = 0$.
Also, $\mu(I,J) = -\sum_{I \subseteq K \subsetneq J} \mu(I,K)$, if $I \subsetneq J$, so by induction,

$$\mu(I,J) = -\sum_{I \subseteq K \subsetneq J} (-1)^{|K|-|I|}.$$

There are $\binom{|J|-|I|}{|K|-|I|}$ subsets of size $|K|$ which contain I. So,

$$\mu(I,J) = -\sum_{i=|I|}^{|J|-1} (-1)^{i-|I|} \binom{|J|-|I|}{i-|I|}.$$

Changing the index of summation to $j = i - |I|$, we find,

$$\begin{aligned} \mu(I,J) &= -\sum_{j=0}^{|J|-|I|-1} (-1)^j \binom{|J|-|I|}{j} \\ &= -\sum_{j=0}^{|J|-|I|} (-1)^j \binom{|J|-|I|}{j} + (-1)^{|J|-|I|} \\ &= -(1-1)^{|J|-|I|} + (-1)^{|J|-|I|} \\ &= (-1)^{|J|-|I|}. \end{aligned}$$

This completes the proof that (8) is the Möbius function of P.

7.3.1 Exercises with Answers

1. Let P be the poset $\{a,b,c,d,e,f\}$ with $\le$ defined by $a \le b \le d \le f$, $a \le c \le e \le f$, $b \le e$. Write down the Hasse diagram for this poset and calculate its Möbius function. If $t(x) = \sum_{y \le x} e(y)$ for all $x \in P$, and t is defined by $t(a) = 1$, $t(b) = -1$, $t(c) = 0$, $t(d) = 3$, $t(e) = 2$, $t(f) = 4$, find E.
[$E = (1,-2,-1,4,4,-2)$]

7.4 Applications

We have seen in Chapter 3 very powerful methods for counting circular patterns. In this section we give another method, based on finding a relationship between linear and circular arrangements and then inverting.

Suppose we want to determine the number of circular patterns that can be made with n beads of c possible colours. We do not distingiush between patterns which are the same except for cyclic rotations, e.g.,

$$\begin{array}{ccc} & R & \\ & \circ & \\ B\circ & & \circ B \\ & \circ & \\ & B & \end{array} = \begin{array}{ccc} & B & \\ & \circ & \\ B\circ & & \circ R \\ & \circ & \\ & B & \end{array}$$

First, the number of rows (linear patterns) of n beads of c colours is c^n.

On the other hand, a circular pattern of n beads can be *cut* at n possible places and stretched out into a linear pattern. So cicular patterns give rise to up to n linear patterns. e.g.,

$$\begin{array}{ccc} & R & \\ & \circ & \\ B\circ & & \circ B \\ & \circ & \\ & B & \end{array} \mapsto \begin{cases} RBBB \\ BRBB \\ BBRB \\ BBBR \end{cases} \qquad \text{but,} \qquad \begin{array}{ccc} & R & \\ & \circ & \\ B\circ & & \circ B \\ & \circ & \\ & R & \end{array} \mapsto \begin{cases} RBRB \\ BRBR \end{cases}$$

Definition. The *period* d of a circular patter is the number of distinct linear patterns it yields.

Observe that if $d < n$ then $d|n$ and each linear pattern contains $\frac{n}{d}$ identical blocks of length d.

Let $M(n,d)$ denote the number of circular patterns of length n and period d. Then the total number of circular patterns is,

$$\sum_{d|n} M(n,d).$$

The circular patters of period d yield $M(n,d)$ linear pattern, so the total number of linear patterns can be counted in two ways to give the identity,

$$c^n = \sum_{d|n} dM(n,d). \tag{9}$$

To find $M(n,n)$, we just invert (9).

$$\begin{aligned} n(M(n,n) &= \sum_{d|n} \mu(d,n)c^d \\ &= \sum_{d|n} \bar{\mu}(\frac{n}{d})c^d, \end{aligned} \tag{10}$$

where μ and $\bar{\mu}$ are calculated as in Example 9.

In order to calculate the total number of circular patterns, we need only one more observation, that

$$M(n,d) = M(d,d), \tag{11}$$

for all d, n, such that $d|n$.

Hence the total number of circular arrangements of n beads of c colours is, by (10) and (11),

$$\sum_{d|n} M(n,d) = \sum_{d|n} M(d,d) = \sum_{d|n} \frac{1}{d} \sum_{k|d} \bar{\mu}(\frac{d}{k}) c^k. \qquad (12)$$

Example 11. find the number of circular patters of six beads of three colours.

$$M(6,1) = M(1,1) = \frac{1}{1}\sum_{d|1} \bar{\mu}(\frac{1}{d})c^d = \bar{\mu}(1) = 3,$$
$$M(6,2) = M(2,2) = \frac{1}{2}\sum_{d|2} \bar{\mu}(\frac{2}{d})c^d = \frac{1}{2}(c^2 - c) = 3,$$
$$M(6,3) = M(3,3) = \frac{1}{3}\sum_{d|3} \bar{\mu}(\frac{3}{d})c^d = \frac{1}{3}(c^3 - c) = 8,$$
$$M(6,6) = \frac{1}{6}\sum_{d|6} \bar{\mu}(\frac{6}{d})c^d = \frac{1}{6}(c^6 - c^3 - c^2 + c) = 116.$$

So, by (12), the total number of circular patterns is,

$$\sum_{d|6} M(6,d) = 130.$$

Inclusion-Exclusion

Let X be a finite set and $A_1, A_2, \ldots, A_n$ be subsets of X.
Formulas (**??**) and (**??**) of Chapter 2 are ways of writing the Inclusion-Exclusion Principle. Let us see how this principle is obtainaed using Möbius functions.
The crucial poset is not connected to X, but to the set of *subscripts* of the subsets A_i. Let P be the poset of all subsets of $[n] = \{1, 2, 3, \ldots, n\}$. We define functions,

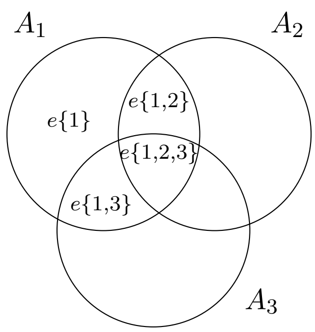
$$s, e \colon P \to \mathbb{R},$$

as follows

$$s(I) = \Big|\bigcap_{i\in I} A_i\Big|,$$

and $e(I)$ is the number of elements of $\bigcap_{i\in I} A_i$, which lie in no subset A_j with $j \notin I$.
Hence, for example, if $n = 3$ (see diagram),

$$s\{1\} = e\{1\} + e\{1,2\} + e\{1,3\} + e\{1,2,3\}.$$



In general,

$$s(I) = \sum_{I \subseteq J} e(J). \tag{13}$$

Inverting (13), we get from (8),

$$e(I) = \sum_{I \subseteq J} \mu(I, J) s(J) = \sum_{I \subseteq J} (-1)^{|J|-|I|} s(J). \tag{13}$$

In particular,

$$e(\emptyset) = \sum_{\emptyset \subseteq J} (-1)^{|J|} s(J),$$

is just the Inclusion-Exclusion principle with $e_0 = e(\emptyset)$ and $s_j = \sum_{|J|=j} s(J)$.

m-Compositions

An *m-composition* of $n \in \mathbb{N}$, or an *ordered partition* of n into m parts, is an ordered m-tuple $(k_1, k_2, \ldots, k_m)$, where the k_i's are positive integers and $k_1 + k_2 + k_3 + \cdots + k_m = n$. If we consider n objects in a row, we can obtain an m-composition of n by inserting $m-1$ markers in the $n-1$ spaces between the objects (k_1 will be the number of objects to the left of the first marker, k_2 the number of objects between the first two markers, and so on). Clearly each m-composition arises in this way. So the total number of m-compositions is $\binom{n-1}{m-1}$.

Two m-compositions are said to be *equivalent* if one is a cyclic permutation of the other. For example, $(1,2,3) \equiv (3,1,2) \equiv (2,3,1)$ but $(1,2,3) \not\equiv (1,3,2)$. Our object will be to calculate the number of non-equivalent m-compositions of n using Möbius inversion. Non-equivalent compositions occur in coding theory as a way of distinguishing codewords of different weights in codes generated by certain types of matrices.

If the m-composition $(k_1, k_2, \ldots, k_m)$ is equivalent to $d-1$ other m-compositions of n, we say that it has *period* d. Denote the number of m-compositions of n of period d by $c_d(m, n)$. If an m-composition of n has period d, then it consists of an ordered d-tuple of positive integers, say $I_1, I_2, \ldots, I_d$ repeated $\frac{m}{d}$ times. Hence $c_d(m, n) = 0$ if d does not divide m. On the other hand, if $t = l_1 + l_2 + \cdots + l_n$, then $\frac{m}{d} t = n$ and so $t = \frac{nd}{m} \in \mathbb{N}$. It follows that,

$$c_d(m, n) = c_d(\frac{nd}{m}, d) \tag{15}$$

From each l-composition of $\frac{nl}{m}$ of period l, we obtain l equivalent l-compositions. Hence the total number of non-equivalent d-compositions of $\frac{nd}{m}$ of period l is from (15),

$$c_l(\frac{nd}{m}, d) = c_l\Big(\frac{\frac{nd}{m} - l}{d}, l\Big) = c_l(\frac{nd}{m}, l).$$

So the total number of d-compositions of $\frac{nd}{m}$ is,

$$\binom{\frac{nd}{m} - 1}{d-1} = \sum_{l|d} l c_l(\frac{nd}{m}, d) = \sum_{l|d} l c_l(\frac{nl}{m}, l). \tag{16}$$

For details of this application, see Razen, R. Seberry, J., Wehrhahn, K., "*Ordered Partitions and Codes Generated by Circulant Matrices*", J. Combinatorial Theory Series A (1979), 333-341. A species interpretation of m-compositions is given in Unger, W., Wehrhahn, K., "*Species of Ordered Partitions*", Ars. Combinatoria 21-A (1986), 141-146.

Inverting (**??**) now gives us the number of m-compositions of n of period d.

$$dc_d(\frac{nd}{m}, d) = \sum_{l|d} \mu(l,d) \binom{\frac{nl}{m} - 1}{l-1} = \sum_{l|d} \mu(l,d) \binom{\frac{nd}{ml} - 1}{\frac{d}{l} - 1}. \tag{17}$$

Since $\binom{p-1}{q-1} = \frac{q}{p}\binom{p}{q}$, we get,

$$c_d(\frac{nd}{m}, d) = \frac{1}{d} \sum_{l|d} \mu(1,l) \frac{d}{l} \frac{ml}{nd} \binom{\frac{nd}{m}}{\frac{d}{l}} = \frac{m}{nd} \sum_{l|d} \bar{\mu}(l) \binom{\frac{nd}{ml}}{\frac{d}{l}}. \tag{18}$$

Now let $c(n,m)$ be the total number of non-equivalent m-compositions of n. Then,

$$c(n,m) = \sum_{d|m} c_d(n,m) = \sum_{d|m} c_{\frac{m}{d}}(n,m)$$

$$\sum_{d|m} \frac{d}{n} \sum_{k|d} \frac{m}{d} \bar{\mu}(k) \binom{\frac{n}{kd}}{\frac{m}{d}}. \tag{19}$$

If $k = p_i$ the contribution is,

$$\begin{aligned} \sum_{\substack{d|m \\ p_i|\frac{m}{d}}} \frac{d}{n} \bar{\mu}(p_i) \binom{\frac{n}{p_i d}}{\frac{m}{p_i d}} &= -\sum_{dp_i|m} \frac{d}{n} \binom{\frac{n}{p_i d}}{\frac{m}{p_i d}} \\ &= -\sum_{l|m} \frac{l}{p_i n} \binom{\frac{n}{l}}{\frac{m}{l}}. \end{aligned}$$

Similarly, if $k = p_i p_j$ the contribution to $c(n,m)$ is,

$$\sum_{l|m} \frac{l}{p_i p_j n} \binom{\frac{n}{l}}{\frac{m}{l}},$$

and so on. Hence we finally get,

$$\begin{aligned} c(n,m) &= \frac{1}{n} \sum_{l|m} l \Big(1 - \sum_i \frac{1}{p_i} + \sum_{i,j} \frac{1}{p_i p_j} - \sum_{i,j,k} \frac{1}{p_i p_j p_k} + \cdots + (-1)t \frac{1}{p_1 p_2 \cdots p_l} \Big) \binom{\frac{n}{l}}{\frac{m}{l}} \\ &= \frac{1}{n} \sum_{l|m} \phi(l) \binom{\frac{n}{l}}{\frac{m}{l}} \qquad \text{from equation (4) of Chapter 2.} \end{aligned}$$

7.4.1 Exercises with Answers

1. Consider the circular arrangements of sixteen beads, in which beads can be colored red, blue or green.

(i) Write out the distinct arrangements of periods 1, 2 and 4.

(ii) Determine the total number of such arrangements. [2, 690, 844]

2. Find the number of non-equivalent 4-compositions of 6 and 8. [3;10]