6 Finite operator calculus

We have seen in Chapter 4 §1 the close connection between the polynomial sequences, $\{x^{}$ and $B_n(x)$, of standard and Bernoulli polynomials respectively and the derivative operator D. In Chapter 2 §5 we saw a similar relationship between the sequence $\{x_{(n)}\}$ and the forward difference operator Δ. In this chapter we will see that this is part of a general pattern. That, for example, many polynomial sequences satisfy a binomial identity, like,

$$(x+y)_{(n)} = \sum_{k\geq 0} \binom{n}{k} x_{(k)} y_{(n-k}.$$

Many other combinatorial identities arise from the interrelationship between operators and polynomial sequences, for example, we will be able to prove Abel's famous generalization of the binomial theorem,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x_k\alpha)^{k-1}(y+k\alpha)^{n-k}.$$

The formulation we give here was developed mainly by Gian-Carlo Rota and collaborators, in a sequence of articles entitled *The Foundations of Combinatorics*.

6.1 Polynomial operators

The operators D and Δ belong to the general class of *polynomial operators*. More precisely, we have:

Definition. Let $\mathbb{R}[x]$ be the vector space of all real polynomials in x. A *polynomial operator* is a linear transformation

$$Q \colon \mathbb{R}[x] \to \mathbb{R}[x].$$

Example 1. Here are some examples of polynomial operators.

(1) The *identity operator* 1 defined by

$$1 \colon p(x) \mapsto p(x).$$

(which leaves each polynomial as it is).

(2) The *shift operators* E^a defined for each $a \in \mathbb{R}$ by

$$E^a \colon p(x) \mapsto p(x+a).$$

(which shift each polynomial a units to the left).

[We often write E for the shift operator E^1.]

(3) The *multiplication operator* x defined by

$$x \colon p(x) \mapsto xp(x).$$

(4) The *derivative operator* D defined by

$$D \colon p(x) \mapsto \frac{d}{dx} p(x).$$

Specifically, see III, *The Theory of Binomial Enumeration*, Ronald Mullin and Rota, Graph Theory and Applications, 167-213, 1970; and VIII, *Finite Operator Calculus*, D. Kahaner, A. Odlyzko, and Rota, Journal of Mathematical Analysis and Applications, 684-760, 1973.

(5) The *forward difference operator* Δ defined by

$$\Delta\colon p(x) \mapsto p(x+1) - p(x).$$

(6) The *backward difference operator* ∇ defined by

$$\nabla\colon p(x) \mapsto p(x) - p(x-1).$$

Note. The most common way to define an operator is to specify its values on a basis of $\mathbb{R}[x]$. The most natural bases are the polynomial sequences $\{p_n(x)\}$, where $\deg(p_n(x)) = n$, for $n = 1, 2, 3, \ldots$, and $p_0(x)$ is a nonzero constant. Thus, if $\{s_n(x) \mid n \text{ a positive integer}\}$ is any set of polynomials, the correspondence

$$p_n(x) \mapsto s_n(x), \qquad n = 0, 1, 2, 3, \ldots,$$

extends linearly to define a polynomial operator. [This means that

$$\sum_{k=0}^{n} a_k p_k(x) \mapsto \sum_{k=0}^{n} a_k s_k(x),$$

for $a_n \in \mathbb{R}$.]

Example 2.

(1) $x^n \mapsto nx^{n-1}$ for $n = 0, 1, 2, 3, \ldots$,

defines the derivative operator $D = \dfrac{d}{dx}$.

(2) $x_{(n)} \mapsto nx_{(n-1)}$, for $n = 0, 1, 2, 3, \ldots$,

defines the forward difference operator Δ.

Remark. (Combinations of Operators.) We can combine operators in the usual ways, common to all linear transformations. Thus $P + Q$, PQ denote the *addition* and *composition*, respectively of operators P and Q. The operator λP is the *scalar multiple* of the real number λ and the operator P.

Example 3.

(1) $\Delta = E - 1$.

(2) $E^a E^b = E^{a+b}$, for any real numbers a and b.

6.1.1 Exercises with Answers

1. Let ∇ be defined by

$$\nabla\colon f(x) \mapsto f(x) - f(x-1), \qquad \text{for } x \in \mathbb{R}[x].$$

Express ∇ in terms of the operators 1 and E^a. [$1 - E^{-1}$]

2. Show that, for $f(x) \in \mathbb{R}[x]$,

$$\nabla^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-k).$$

6.2 Differential operators

Because of their special relationship with the sequences $\{x^n\}$ and $\{x_{(n)}\}$, respectively, the derivative and forward difference operators are not typical of polynomial operators. In this section we will look at other operators associated with polynomial sequences.

Definition.. Given a sequence $\{p_n(x)\}$ the unique operator Q defined by,

$$Qp_0(x) = 0, \tag{i}$$
$$Qp_n(x) = np_{n-1}(x), \tag{ii}$$

is called the *basis operator* for the sequence $\{p_n(x)\}$. Conversely, given a polynomial operator Q, any sequence $\{p_n(x)\}$ satisfying conditions (i) and (ii) is called a *basis sequence* for Q.

Note. Basis operators reduce the degree of each polynomial by exactly one. The same basis operator has many basis sequences associated with it. For example, D is a basis operator for the sequence $\{x^n\}$ of standard polynomials, but also for the Bernoulli polynomials $\{B_n(x)\}$ as we have seen in Chapter 4. In fact any sequence of the form $\{(x+a)^n\}$, where $a \in \mathbb{R}$, is a basis sequence for D. This state of affairs is unsatisfactory, so we associate with each basis operator a definite basis sequence. To do this we limit ourselves to the class of *diferential operators*, and we generalize the standard polynomials $\{x^n\}$ to the class of *normalized polynomial sequences*. We will see, in Proposition 2.1, that there is a one-to-one correspondence between differential operators and normalized sequences.

Definition. A polynomial operator Q is a *differential operator*, or of *differential type*, if

$$\deg Qp(x) = \deg p(x) - 1, \qquad \text{for } p(x) \in \mathbb{R}[x],\ p(x) \neq 0.$$

In particular $Qa = 0$ for a nonzero constant.

Convention. Define $\deg 0 = -1$.

Definition. A polynomial sequence $\{p_n(x)\}$ is said to be *normalized* if

$$\begin{array}{ll} \text{(i)} & p_0(x) = 1, \\ \text{(ii)} & p_n(0) = 0, \quad \text{for } n \geq 1. \end{array}$$

The characterization of the standard polynomials $\{x^n\}$ in Chapter 4 §1, generalizes in the following way.

Proposition 6.1.

(i) Any polynomial sequence possesses a unique basis operator, which is of differential type.

(ii) Any differential operator Q has a unique normalized basis sequence $\{p_n(x)\}$.

Proof. By definition *any* polynomial sequence has a unique basis operator which is always of differential type, so part (i) is clear.

To prove (ii), we use induction on n to show that there exists a unique polynomial $p_n(x)$ of degree n satisfying the conditions

(a) $p_0(x) = 1$,

(b) $Qp_n(x) = np_{n-1}(x)$,

(c) $p_n(0) = 0$, for $n \geq 1$.

Assume that $p_k(x)$ has been defined for $k < n$, satisfying (b) and (c) and that they are uniquely determined. Any polynomial $p(x)$ of degree n can be written in the form

$$p(x) = ax^n + \sum_{k=0}^{n-1} b_k p_k(x), \qquad \text{with } a \neq 0,$$

since $p_k(x)$ has degree k. Thus,

$$Qp(x) = aQx^n + \sum_{k=0}^{n-1} b_k Q p_k(x) = aQx^n + \sum_{k=0}^{n-1} b_k k p_{k-1}(x).$$

Since Qx^n has degree $n-1$ and $p_j(x) has degree j, there exist unique constants, a_1, b_1, b_2, \ldots, b_{n-1}$, such that $p(x)$ satisfies,

$$Qp(x) = np_{n-1}(x).$$

[i.e. the space of polynomials of degree at most $n-1$ has dimension n and basis

$$Qx^n, \quad p_0(x), \quad 2p_1(x), \quad \ldots, \quad (n-1)p_{n-2}(x),$$

so $np_{n-1}(x)$ can be uniquely expressed in terms of this basis.] This determines $p(x)$ except for the constant b_0. TO satisfy the condition that $p(0) = 0$ we need $b_0 = 0$. Hence there is a unique polynomial $p(x)$ of degree n satisfying

$$Qp(x) = np_{n-1}(x) \qquad \text{and} \qquad p(0) = 0.$$

We set $p_n(x) = p(x)$. □

Corollary 6.2. *The correspondence between basis operator and normalized basis sequence determines a bijection between the set of differential operators and the set of normalized sequences.*

Example 4.

(i) D is the basis operator of the sequence $\{x^n\}$.

(ii) Δ is the basis operator of the sequence $\{x_{(n)}\}$.

6.2.1 Exercises with Answers

1. (i) Find the normalized basis sequence for $\nabla = 1 - E^{-1}$. [$\{x^{(n)}\}$]

(ii) Find the basis operator, in simple form, for the normalized sequence of *Abel polynomials*,

$$p_n(x) = \begin{cases} x(x-an)^{n-1}, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

[The *Abel operator* $DE^n = E^n D$]

6.3 Formulas of Maclaurin type

We show in this section that the Maclaurin formula, geralizes to any differential operator and its associated normalized basis sequence.

Theorem 6.3. *Let $\{p_n(x)\}$ be a normalized sequence with basis operator Q. Then for any polynomial $f(x) \in \mathbb{R}[x]$ we have*

$$f(x) = \sum_{k\geq 0} \frac{[Q^k f(x)]_{x=0}}{k!} p_k(x). \tag{1}$$

Proof. We first show that the theorem holds on a basis for $\mathbb{R}[x]$, namely the sequence $\{p_n(x)\}$ itself. We compute, for $f(x) = p_n(x)$:

$$\begin{aligned} Qp_n(x) &= np_{n-1}(x), \\ Q^2 p_n(x) &= n(n-1)p_{n-2}(x), \\ &\vdots \\ Q^k p_n(x) &= n_{(k)} p_{n-k}(x). \end{aligned}$$

If follows that

$$[Q^k p_n(x)]_{x=0} = n_{(k)} p_{n-k}(0) = \begin{cases} 0, & \text{if } n \neq k, \\ n!, & \text{if } n = k. \end{cases}$$

Hence

$$\sum_{k\geq 0} \frac{Q^k p_n(x)]_{x=0}}{k!} p_k(x) = \frac{n!}{n!} p_n(x) = p_n(x).$$

So the theorem holds for $f(x) = p_n(x)$.

Now suppose

$$f(x) = \sum_{n\geq 0} a_n p_n(x).$$

Then

$$\begin{aligned} Q^k f(x) &= \sum_{n\geq 0} a_n n_{(k)} p_{n-k}(x), \\ [Q^k f(x)]_{x=0} &= a_k k_{(k)} = a_k k!, \\ \text{and,} \qquad a_n &= \frac{[Q^n f(x)]_{x=0}}{n!}, \end{aligned}$$

which completes the proof. □

Applications

Example 5. If $Q = D$ and $\{p_n(x)\} = \{x^n\}$ then

$$f(x) = \sum_{k\geq 0} \frac{[D^k f(x)]_{x=0}}{k!} x^k = \sum_{k\geq 0} \frac{f^{(k)}(0)}{k!} x^k,$$

which is just *Maclaurin's expansion* for polynomials.

Example 6. If $Q = \Delta$ and $\{p_n(x)\} = \{x_{(n)}\}$ then

$$f(x) = \sum_{k\geq 0} \frac{[\Delta^k f(x)]_{x=0}}{k!} x_{(k)},$$

which is *Newton's expansion* important in numerical analysis.

Remark. We can also view formula (1) as giving a partial answer to the following problem.

The Connection Coefficient Problem. Given two polynomial sequences $\{p_n(x)\}$ and $\{q_n(x)\}$, determine the connection coefficients $a_{n,k}$ defined by

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} q_k(x).$$

Example 7. If $Q = \Delta$, $p_n(x) = x^n$, $q_k(x) = x_{(k)}$.
Then,

$$x^n = \sum_{k=0}^{n} [\Delta^n x^n]_{x=0} \frac{x_{(k)}}{k!}.$$

This gives the identity for Stirling numbers of the second kind,

$$S(n,k) = \frac{1}{k!}[\Delta^n x^n]_{x=0} = \frac{1}{k!}[(E+I)^k x^n]_{x=0} = \frac{1}{k!}\sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} i^n.$$

6.3.1 Exercises

1. Use Newton's formula to show that

$$x^{(n)} = \sum_{k\geq 0} \frac{n!}{k!}\binom{n-1}{k-1} x_{(k)}.$$

2. Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be normalized sequences with basis oeprators P and Q, respectively. If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, show that

$$v_n = \sum_{k=0}^{n} \frac{[Q^k p_n(x)]_{x=0}}{k!} u_k \qquad \text{if and only if} \qquad u_n = \sum_{k=0}^{n} \frac{[P^k q_n(x)]_{x=0}}{k!} v_k.$$

6.4 Binomial sequences

In the last section we were able to easily generalize Maclaurin's formula. When we try to do the same thing for Taylor's formula there is an interesting complication which arises. It concerns the following question.

Question. Given a differential operator Q with basis sequence $\{p_n(x)\}$ and any polynomal $f(x)$, is it true that

$$f(x+a) = \sum_{k\geq 0} \frac{[Q^k f(x)]_{x=a}}{k!} p_k(x)? \tag{2}$$

[In other words, is there a Taylor type generalization of Theorem 3.1?]

Answer. Not quite! Not for Q an arbitrary differential operator.

For suppose that (2) did hold for all $f(x)$. Then taking $f(x) = p_n(x)$ we would get

$$p_n(x+a) = \sum_{k\geq 0} \frac{[Q^k f(x)]_{x=a}}{k!} p_k(x) = \sum_{k\geq 0} \frac{n_{(k)}}{k!} p_{n-k}(a) p_k(x)$$
$$= \sum_{k\geq 0} \binom{n}{k} p_k(x) p_{n-k}(a),$$

for all values of a. That is, for (2) to hold, $\{p_n(x)\}$ would need to be of *binomial type*, and we will soon see that normalized sequences need not be of binomial type.

Definition. A polynomial sequence $\{p_n(x)\}$ is of *binomial type* or a *binomial sequence* if

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \tag{3}$$

for $n = 0, 1, 2, \ldots$.

Note. A normalized sequence $\{p_n(x)\}$ need not be of binomial type.

Example 8. Take the sequence

$$\{1,\ x,\ 5x^2,\ x^3,\ \ldots\},$$

which is clearly normalized. If this were a binomial sequence then we would by definition have the identity

$$p_2(x+y) = \binom{2}{0} p_0(x)p_2(y) + \binom{2}{1} p_1(x)p_1(y) + \binom{2}{2} p_2(x)p_0(y).$$

Or,

$$5(x+y)^2 = 5y^2 + 2xy + 5x^2,$$

which is not true.

However the converse is true as we see in the following.

Lemma 6.4. *Any binomial sequence is normalized.*

Proof. Consider a binomial sequence $\{p_n(x)\}$. We have

$$p_n(x) = p_n(x+0) = \sum_{k=0}^{n} p_k(x) p_{n-k}(0) \tag{4}$$
$$= \binom{n}{0} p_n(0)p_0(x) + \binom{n}{1} p_{n-1}(0)p_1(x) + \cdots + \binom{n}{n} p_0(0)p_n(x)$$

for $n = 0, 1, 2, \ldots$.

Comparing coefficients in the polynomial identity (4), we find

$$p_n(x) = p_0(0) = 1 \quad \text{and} \quad p_n(0) = 0, \quad \text{for } n \geq 0.$$

$\square$

Remark. Though most normalized sequences are not of binomial type, many important sequences are. The sequence of standard polynomials $\{x^n\}$ is the example which inspired the name. As we will see, other examples include the sequences $\{x_{(n)}\}$, $\{x^{(n)}\}$, $\{e_n(x)\}$, where $x^{(n)} = x(x+1)\cdots(x+n-1)$, $e_n = \sum_{k=0}^{n} S(n,k)x^k$ are the *rising factorial* and the *exponential polynomials* respectively.

Let us return to the question at the beginning of this section. From Theorem 3.1,

$$f(x+a) = \sum_{k\geq 0} \frac{[Q^k f(x+a)]_{x=0}}{k!} p_k(x) \tag{5}$$

so the original question reduces to deciding when

$$[Q^k f(x)]_{x=a} = [Q^k f(x+a)]_{x=0}. \tag{6}$$

To settle this, we first define the *linear functional*, L_n, called the *evaluation at* a, by

$$L_a p(x) = p(a).$$

In particular write $L = L_0$. Then the left-hand side of (6) can be written as

$$L_a Q^k f(x) = LE^a Q^k f(x),$$

since $l_a = LE^a$. The right-hand side of (6) is $LQ^k E^a f(x)$. Hence (6) holds if and only if

$$QE^a = E^a Q, \tag{7}$$

for all $a \in \mathbb{R}$. A *delta operator* is a shift invariant differential operator.

We can now characterize sequences of binomial type, in terms of their basis operators.

Theorem 6.5. *A normalized poylnomial sequence $\{p_n(x)\}$ is of binomial type if and only if its basis operator is a delta operator.*

Proof. We have already seen that if Q is a delta operator then (6) holds and so the sequence is of binomial type. Conversely, suppose that $\{p_n(x)\}$ is a binomial sequence with Q its basis operator. Then Q is of differential type and it suffices to show that Q is shift invariant. In fact it suffices to show that Q is shift invariant on the basis $\{p_n(x)\}$. Let's do the computation:

$$\begin{aligned}
QE^a p_n(x) = Qp_n(x+a) &= Q\sum_{k\geq 0}\binom{n}{k} p_k(x)p_{n-k}(a) \\
&= \sum_{k\geq 0}\binom{n}{k} kp_{k-1}(x)p_{n-k}(a) \qquad \left(\text{since} \quad k\binom{n}{k} = n\binom{n-1}{k-1}\right) \\
&= n\sum_{k\geq 1}\binom{n-1}{k-1} p_{k-1}(x)p_{n-k}(a) \\
&= n\sum_{l=0}^{n-1}\binom{n-1}{l} p_l(x)p_{n-1-l}(a) \qquad (\text{setting } l = k-1) \\
&= np_{n-1}(x+a) \\
&= E^a Q p_n(x).
\end{aligned}$$

Hence QE^a and E^aQ agree on a basis and so agree for all of $\mathbb{R}[x]$, which concludes the proof that Q is a delta operator. $\square$

Remark. The normalized basis sequence of a delta operator Q will often be called the *binomial sequence* of Q, to emphasize its special character.

Example 9. D, Δ, ∇ are clearly delta operators and so is DE^a. Hence we have the following identitites.

$$\begin{aligned}
(x+y)^n &= \sum_{k\geq 0} \binom{n}{k} x^k y^{n-k}, \\
(x+y)_{(n)} &= \sum_{k\geq 0} \binom{n}{k} x_{(k)} y_{(n-k)}, \\
(x+y)^{(n)} &= \sum_{k\geq 0} \binom{n}{k} x^{(k)} y^{(n-k)}, \\
(x+y)(x+y-na)^{n-1} &= \sum_{k\geq 0} \binom{n}{k} x(x-ka)^{k-1} y(y-(n-ka)^{n-k-1}.
\end{aligned}$$

Since (6) holds provided Q is shift invariant, we finally obtain from (5) the following *Taylor-type formula* for poylnomials:

Theorem 6.6. *If Q is a delta operator with binomial sequence $\{p_n(x)\}$ then*

$$f(x+a) = \sum_{k\geq 0} \frac{[Q^k f(x)]_{x=a}}{k!} p_k(x), \tag{8}$$

for all $f(x) \in \mathbb{R}[x]$ and $a \in \mathbb{R}$.

There is another useful characterization of delta oeprators.

Lemma 6.7. *A shift invariant operator Q is a delta operator if and only if Qx is a non-zero constant.*

Proof. If Q is a delta operator, then Qx is certainly a non-zero constant. Conversely, suppose that $Qx = c \neq 0$ for some constant c. To prove that Q is differential and hence a delta operator it suffices to show that Qx^n is of degree $n-1$.

Let $r(x) = Qx^n$. Since Q is shift invariant we have

$$Qe^a x^n = E^a Q x^n,$$

for $a \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$. Thus,

$$r(x+a) = Q(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k Q x^{n-k}. \tag{9}$$

Evaluating both sides of (9) at $x = 0$ we get

$$r(a) = \sum_{k=0}^{n} \binom{n}{k} a^k [Qx^{n-k}]_{x=0}.$$

Hence

$$r(x) = \sum_{k=0}^{n} \binom{n}{k} [Qx^{n-k}]_{x=0} x^k.$$

Moreover, the coefficient of x^n is $Q1$, and the coefficient of x^{n-1} is

$$\binom{n}{1}[Qx]_{x=0} = nc \neq 0.$$

It remains to show that $Q1 = 0$. But $QE^a x = Q(x+a) = c + Qa$ and $E^a Qx = e^a c = c$, so $Qa = 0$ for all a, by shift invariance. ☐

Example 10. (Abel's Generalization of the Binomial Theorem.) Let us expand $f(x) = x^2$ in terms of the Abel polynomials using Theorem 4.3.

$$(x+b)^n = \sum_{k\geq 0} \frac{1}{k!}[(DE^a)^k x^n]_{x=b} x(x-ka)^{k-1}.$$

But,

$$\begin{aligned}(DE^a)^k x^n &= D^k(E^{ak}x^n) = E^{ak}(D^k x^n) \\ &= E^{ak} n_{(k)} x^{n-k} = n_{(k)}(x+ak)^{n-k}.\end{aligned}$$

Hence,

$$[(DE^a)^k x^n]_{x=b} = n_{(k)}(b+ak)^{n-k} = k!\binom{n}{k}(b+ak)^{n-k},$$

which gives,

$$(x+b)^n = \sum_{k\geq 0}\binom{n}{k} x(x-ka)^{k-1}(b+ak)^{n-k}.$$

Since this holds for all $b \in \mathbb{R}$, we get *Abel's generalization of the binomial theorem*,

$$(x+y)^n = \sum_{k\geq 0}\binom{n}{k} x(x-ka)^{k-1}(y+ak)^{n-k}.$$

For example, if $x = 1$, $y = 0$, $a = 1$, we have the special case,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k-1}(k-1)^{k-1}k^{n-k} = 1.$$

6.4.1 Exercises

1. Find the polynomial of degree three which passes through the points $(2,15)$, $(3,21)$, $(4,19)$, (5.3). $[3 + 5x^2 - x^3]$

2. How could you use (8) to find the unique quadratic curve through the points $(2,-1)$, (4.0), $(6,3)$? $[\frac{1}{4}x^2 - x]$

3. Establish Abel's identity:

$$(x+y-na)^n = \sum_{k=0}^{n}\binom{n}{k} x(x-ka)^{k-1}(y-(n-k)a)^{n-k},$$

and its corollary:

$$\sum_{k=1}^{n}\binom{n}{k}(k-1)^{k-1}(n-k)^{n-k} = n^n - (n-1)^n, \qquad \text{where } 0^0 = 1.$$

6.5 The first expansion theorem

The delta operators of the last section are shift invariant. What role do they play in the set Σ of all shift invariant operators on $\mathbb{R}[x]$?

The set Σ is an algebra with respect to addition and scalar multiplication of operators and the usual composition of operators as product. Examples of shift operators include $I, D, \Delta, nabla, E^a$, and any linear combination of products of these. In fact any formal power series in a delta operator will be shift invariant. For example, let

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \qquad (10)$$

be a formal power series. Then

$$f(D) = \sum_{k=0}^{\infty} a_k D^k \qquad (11)$$

is the operator on $\mathbb{R}[x]$ defined by

$$f(D)x^n = \sum_{k=0}^{\infty} a_k n_{(k)} x^{n-k}.$$

In other words, when $f(D)$ is applied to any polynomial, only a finite number of terms will be different from 0. The remarkable fact is that every shift invariant operator is of the form (11). To prove this we start with two small lemmas.

Lemma 6.8. *Let P and Q be two polynomial operators. Then P and Q are identical if and only if*

$$L_a P = L_a Q,$$

for all $a \in \mathbb{R}$.

Proof. Suppose that $L_a P = L_a Q$ for all $a \in \mathbb{R}$. Let $f \in \mathbb{R}[x]$ be an arbitrary polynomial. We want to show that $Pf = Qf$.

Suppose

$$Pf \colon x \mapsto \sum a_k x^k, \quad \text{for } x \in \mathbb{R}, \qquad \text{and} \qquad Qf \colon x \mapsto \sum b_l x^l.$$

Then

$$L_a(Pf) \colon x \mapsto \sum a_k a^k, \qquad \text{and} \qquad L_a(Qf) \colon x \mapsto \sum b_l a^l.$$

Since $L_a(Pf) = L_a(Qf)$ for all a we have that

$$\sum a_k a^k = \sum b_l a^l, \qquad \text{for all } a \in \mathbb{R},$$

and hence

$$Pf = Qf \qquad \text{and so} \quad P = Q.$$

□

Lemma 6.9. *If P and Q are shift invariant, then $P = Q$ if and only if $LP = LQ$.*

Proof. Suppose $LP = LQ$. Then

$$LPE^a = LQE^a$$

and so

$$LE^a P = LE^a Q,$$

for all $a \in \mathbb{R}$. Thus, by Lemma 6.8, $L_a P = L_a Q$ for all $a \in \mathbb{R}$ and $P = Q$. □

Theorem 6.10. ***(First expansion Theorem.)*** *Let P be a shift invariant operator. Then*

$$P = \sum_{k\geq 0} \frac{a_k}{k!} D^k,$$

where $a_k = [Px^k]_{x=0}$. Conversely, any expansion

$$\sum_{k\geq 0} \frac{a_k}{k!} D^k$$

is shift invariant.

Proof. Let $Q = \sum_{k\geq 0} \frac{a_k}{k!} D^k$. It suffices, by Lemma 5.2, to show that LP and LQ agree on the basis $\{x^n\}$ of $\mathbb{R}[x]$. But

$$Qx^n = \sum_{k=0}^{n} \frac{a_k}{k!} n_{(k)} x^{n-k}$$

and so $LQx^n = \dfrac{a_n}{n!} n! = a_n$. But by defintiion $LPx^n = [Px^n]_{x=0} - a_n$. Hence $LP = LQ$ and so $P = Q$. □

In fact we can replace D by any delta operator and $\{x^n\}$ by its binomial sequence to get the following generalization.

Theorem 6.11. *Let P be any shift invariant operator and Q a delta operator with binomial sequence $\{p_n(x)\}$. Then*

$$P = \sum_{k\geq 0} \frac{a_k}{k!} Q^k,$$

where $a_k = [Pp_k(x)]_{x=0}$ for all k.

Example 11.

(i) $[E^a x^n]_{x=0} = (x+a)^n|_{x=0} = a^n$, so

$$E^a = \sum_{k\geq 0} \frac{a_k}{k!} D^k = \exp(aD).$$

In other words,

$$f(x+a) = \sum_{k\geq 0} \sum_{k\geq 0} \frac{f^{(k)}(x)}{k!} a^k$$

for all polynomials $f(x) \in \mathbb{R}$.

[Tihs is Taylor's expansion of $f(x+a)$ about $x = a$.]

(ii) $\Delta = E = I = \exp(D) - I$.

Note. The First Expansion Theorem shows that **no** operator which raises the degree of a polynomial can be shift invariant.

6.5.1 Exercises with Answers

1. Expand E^a in terms of Δ; what do you get if a is a positive integer, say $a = n \geq 1$?
$[E^a = \sum_{k=0}^{n} \binom{n}{k} \Delta^a]$

6.6 Sheffer sequences

We have seen in §3 and §4 how the Maclaurin and Taylor formulas have general analogues for polynomials. In this section look again at the Bernoulli polynoimals and Euler-Maclaurin's summation formula, to see how they generalize.

The Bernoulli Polynomials

By the definition of §1, Chapter 4,

$$DB_{n+1} = (n+1)B_n(x), \qquad \text{for } n \geq 0. \tag{12}$$

By the Fundamental theorem of Calculus, we can integrate to get

$$\begin{aligned} \int_x^{x+1} DB_{n+1}(t)\,dt = B_{n+1}(t)\Big|_x^{x+1} &= B_{n+1}(x+1) - B_{n+1}(x) \\ &= \Delta B_{n+1}(x), \qquad \text{for } n \geq 0. \end{aligned} \tag{13}$$

Definition. Define the *Bernoulli operator* J by,

$$Jp(x) = \int_x^{x+1} p(t)\,dt, \qquad \text{for } p(x) \in \mathbb{R}[x].$$

By (13), we have,

$$JDB_{n+1}(x) = \Delta B_{n+1}(x), \qquad \text{for } n \geq 0.$$

Hence JD and Δ agree on a basis of $\mathbb{R}[x]$ and so we have the operator identity,

$$JD = \Delta. \tag{14}$$

On the other hand, by Lemma 1.2, Chapter 4,

$$\Delta B_{n+1}(x) = (n+1)x^n. \tag{15}$$

So

$$(n+1)x^n = JDB_{n+1}(x) = (n+1)JB_n(x), \qquad \text{for } n \geq 0.$$

In other words,

$$JB_n(x) = x^n, \qquad \text{for } n \geq 0. \tag{16}$$

Remark. From (16), we see that the link between the two basis sequences $\{x^n\}$ and $\{B_n(x)\}$, of the delta operator D, is via the new operator J.

Clearly we need to study J.

Properties of J

(1) J is shift invariant.

For any real number a and $g(x) \in \mathbb{R}[x]$, we have

$$\begin{aligned} JE^a g(x) = Jg(x+a) &= \int_x^{x+1} g(t+1)\,dt \\ &= \int_{x+a}^{x+a+1} g(u)\,du \qquad \text{[substituting } u = t+a\text{]} \\ &= E^a Jg(x). \end{aligned}$$

□

(2) J is invertible.

This is clear from Corollary 6.3, since $J1 = \int_x^{x+1} 1\,dt = 1$. □

We have thus, an alternative characterization of the Bernoulli polynomials,

$$B_n(x) = J^{-1}x^n, \qquad \text{for } n \geq 0. \tag{17}$$

To calculate some Bernoulli polynomials from (17), proceed as follows.

$$\begin{aligned} J1 &= \int_x^{x+1} 1\,dt = 1, \qquad \text{so} \quad J^{-1}1 = B_0(x) = 1; \\ J(x) &= \int_x^{x+1} t\,dt = x + \frac{1}{2}, \qquad \text{so} \quad J^{-1}x = B_1(x) = x - \frac{1}{2}; \\ J(x^2) &= \int_x^{x+1} t^2\,dt = \frac{t^3}{3}\Big|_x^{x+1} = x^2 + x + \frac{1}{3}, \\ \text{and so} \quad J^{-1}(x^2) &= B_2(x) = x^2 - J^{-1}x - J^{-1}\frac{1}{3} = x^2 - x + \frac{1}{6}. \end{aligned}$$

It's time to put these results into a more general setting.

Sheffer sequences

Defintiion. Let P be a delta operator with associated binomial sequence $\{p_n(x)\}$ and S and invertible shift invariant operator. We call $\{Sp_n(x)\}$ a *Sheffer sequence* for P.

Example 12. By (17), $\{B_n(x)\}$ is a Sheffer sequence for D. Also, since E^a is shift invariant and $E^n 1 = 1$, $\{E^n x^n\} = \{(x+a)^n\}$ is a Sheffer sequence for the derivative operator D, for each $a \in \mathbb{R}$.

Properties of Sheffer sequences

The relationship between a delta operator and it Sheffer sequences is very similar to that between the delta operator and its binomial sequence, as we see from the following lemmas.

Lemma 6.12. *Let $\{Sp_n(x)\}$ be a Sheffer sequence for P. Then*

$$\begin{aligned} &P(Sp_n(x)) = nSp_{n-1}(x), \qquad \textit{for } n > 0, \\ &Sp_0(x) = c \neq 0. \end{aligned}$$

Proof.
$$PSp_n(x) = SPp_n(x) = Snp_{n-1}(x) = nSp_{n-1}(x).$$
Also since S is invertible, we have $Sp_n(x) = S1 \neq 0$. □

Lemma 6.13. *Let $\{s_n(x)\}$ be a sequence of polynomials with:*
$$\begin{aligned} Ps_n(x) &= ns_{n-1}(x), \qquad \text{for } n > 0, \\ s_0(x) &= c \neq 0, \end{aligned}$$
where P is a delta operator. Then $\{s_n(x)\}$ is a Sheffer sequence.

Proof. Define $S\colon p_n(x) \to s_n(x)$, where $\{p_n(x)\}$ is the normalized basis sequence for P. S is invertible and it remains to show that S is shift invariant. But
$$SPp_n(x) = S(np_{n-1}(x) = ns_{n-1}(x) = Ps_n(x) = PSp_n(x).$$
Hence $SP = PS$ and so $SP^k = P^kS$ for all integers $k \geq 0$.
By the First Expansion Theorem,
$$E^n = \sum_{k \geq 0} \frac{a_k}{k!} P^k, \tag{18}$$
where $a_k = E^a p_k(x)|_{x=0} = p_k(a)$. Hence $E^aS = SE^a$ for all a and so S is shift invariant. □

Sheffer sequences satisfy the following *Binomial Theorem.*

Theorem 6.14. *Let P be the delta operator with associated binomial sequence $\{p_n(x)\}$. Suppose S is shift invariant and invertible. The the Sheffer sequence $\{Sp_n(x)\} = \{s_n(x)\}$ satisfies the identities:*
$$s_n(x+y) = \sum_{k \geq 0} \binom{n}{k} s_k(x) p_{n-k}(y), \qquad \text{for } n = 0, 1, 2, \ldots. \tag{19}$$

Proof. Since $\{p_n(x)\}$ is a binomial sequence, we have,
$$p_n(x+y) = \sum_{k \geq 0} \binom{n}{k} p_k(x) p_{n-k}(y), \qquad \text{for } n = 0, 1, 2, \ldots.$$
Apply S to both sides, holding y constant, to get
$$\begin{aligned} \sum_{k \geq 0} \binom{n}{k} s_k(x) p_{n-k}(y) &= S(p_n(x+y)) = SE^y(p_n(x)) = E^yS(p_n(x)) = E^y(s_n(x)) \\ &= s_n(x+y), \qquad \text{for } n = 0, 1, 2, \ldots. \end{aligned}$$
□

As a corollary, we see that the general computational formula for the Bernoulli polynomials, Chapter 4, (3), is typical of all Sheffer sequences.

Corollary 6.15. *If $\{p_n(x)\}$ and $\{s_n(x)\}$ are as in Theorem 6.14, then*
$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_k(0) p_{n-k}(x). \tag{20}$$

Proof. Let $x = 0$ in (19), and then replace y by x, in the resulting formula. □

Formula (18) can be rewritten as

$$E^a = \sum_{k\geq 0} \frac{p_k(a)}{k!} P^k, \tag{21}$$

which generalizes to Sheffer sequences in the following way.

Theorem 6.16. *(Second Expansion Theorem.) Let P be a delta operator with binomial sequence $\{p_n(x)\}$ and let $S \in \Sigma$ be invertible with $Sp_n(x) = s_n(x)$. Then for any $a \in \mathbb{R}$ we have*

$$E^a = \sum_{n\geq 0} \frac{s_n(a)}{n!} P^n S^{-1}.$$

Proof. Apply the First Expansion Theorem for $E^a S$.

$$E^a S = \sum_{n\geq 0} \frac{[E^a S p_n(x)]_{x=0}}{n!} P^n = \sum_{n\geq 0} \frac{[E^a s_n(x)]_{x=0}}{n!} P^n = \sum_{n\geq 0} \frac{s_n(a)}{n!} P^n.$$

□

Corollary 6.17.

$$S = \sum_{n\geq 0} \frac{s_n(0)}{n!} P^n.$$

The Euler-Maclaurin Summation Formula

Let us return to the example that lead to the above discussion, the Bernoulli polynomials.

$$B_n(x) = J^{-1} x^n.$$

By Corollary 6.17 we have,

$$I = \sum_{n\geq 0} \frac{B_n(0)}{n!} D^n J. \tag{22}$$

Hence for $f(x) \in \mathbb{R}[x]$, we obtain,

$$f(x) = \sum_{n\geq 0} \frac{B_n(0)}{n!} J D^n f(x). \tag{23}$$

Thus,

$$\begin{aligned} f(x) = &\int_x^{x+1} f(x)\, dx - \tfrac{1}{2}(f(x+1) - f(x)) + \tfrac{1}{6} \cdots \tfrac{1}{2}(f'(x+1) - f'(x)) \\ &+ \sum_{n\geq 3} \frac{B_n(0)}{n!} (f^{(n-1)}(x+1) - f^{(n-1)}(x)), \end{aligned}$$

which leads to the Euler-Maclaurin Summation Formula for polynomials

$$\begin{aligned}\sum_{k=0}^{n} f(k) &= \int_0^{n+1} f(x)\,dx - \tfrac{1}{2}(f(n+1)-f(0)) + \tfrac{1}{12}(f'(n+1)-f'(0)) \\ &\quad + \sum_{n\geq 3} \frac{B_n(0}{n!}(f^{(n-1)}(n+1)-f^{(n-1)}(0)).\end{aligned} \tag{24}$$

Example 12. Find the sum $\displaystyle\sum_{k=0}^{n}(k-2)^3$.

Here $f(x)=(x-2)^3$ and $f'(x)=3(x-2)^3$, $f''(x)=6(x-2)$. Hence,

$$\begin{aligned}\sum_{k=0}^{n}(k-2)^3 &= \int_0^{n+1}(x-2)^3\,dx - \tfrac{1}{2}((n-1)^3+8) + \tfrac{1}{12}(3(n-1)^2-12) \\ &= \Big(\frac{(n-1)(n-2)}{2}\Big)^2 - 9.\end{aligned}$$