5 The Euler-Maclaurin summation formula

In this chapter we will find a precise relationship between summation and integration. A study of the Bernoulli polynomials and numbers that arise here will allow us to prove such famous formulas as,

$$1^p + 2^p + 3^p + \cdots + (n-1)^p = \frac{1}{p+1}\big((n+B)^{p+1} - B^{p+1}\big),$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$

and see how Stirling's formula,

$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n,$$

may be derived.

5.1 Bernoulli numbers and polynomials

Recall first a common characterization for the standard polynomials $\{x^n\}$.
Define $\{P_n(x)\}$ by:

(i)* $P_0(x) = 1$,

(ii)* $P_n'(x) = nP_{n-1}(x)$, for $n \geq 1$,

$$\text{or equivalently, } P_n(x) = \int nP_{n-1}(x)dx,$$

(iii)* $P_n(0) = 0$, for $n \geq 1$.

It is easy to check that $P_n(x) = x^n$ for $n = 0, 1, 2, \ldots$.

Remark. Condition (iii)* is necessary to uniquely determine the polynomials. It says that each of the polynomials, except for $P_0(x)$, passes through the origin.

Defintiion. The *Bernoulli polynomials* $\{B_n(x)\}$ are characterized by the conditions

(i) $B_0(x) = 1$,

(ii) $B_n'(x) = nB_{n-1}(x)$, for $n \geq 1$,

$$\text{or equivalently, } B_n(x) = \int nB_{n-1}(x)dx,$$

(iii) $\int_0^1 B_n(x)dx = 0$, for $n \geq 1$.

Note. Condition (iii) now forces the $B_n(x)$, for $n \geq 1$ to have an average value of 0 over the interval $[0, 1]$.

Definition. The *nth Bernoulli number* B_n is defined by

$$B_n = B_n(0), \quad \text{for } n = 0, 1, 2, \ldots.$$

Example 1.

$$B_1(x) = \int 1B_0(x)dx = x + C, \quad \text{by (ii)}$$

and

$$\int_0^1 b_1(x)dx = 0, \qquad \text{which implies } \tfrac{1}{2}x^2 + Cx\Big|_0^1 = 0, \qquad \text{so } C = -\tfrac{1}{2}.$$

Hence

$$\begin{aligned} B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= \int 2B_1(x)dx = 2\Big(\frac{x^2}{2} - \frac{1}{2}x\Big) + C, \\ &= x^2 - x + C. \end{aligned}$$

Also,

$$\int_0^1 (x^2 - \frac{1}{2}x + C)dx = \frac{x^3}{3} - \frac{x^2}{2} + Cx\Big|_0^1 = 0,$$

hence,

$$C = \frac{1}{6} \quad \text{and} \quad B_2(x) = x^2 - x + \frac{1}{6}.$$

A more efficient formula can be obtained by studying the *Maclaurin series* for $B_n(x)$, namely,

$$B_n(x) = \sum_{k=0}^{\infty} B_n^{(k)}(0)\frac{x^k}{k!}.$$

Then kth derivative of $B_n(x)$ is

$$\begin{aligned} B_n^{(k)}(x) &= n(n-1)(n-2)\cdots(n-k+1)B_{n-k}(x) \\ &= n_{(k)}B_{n-k}(x), \quad \text{for } k = 0, 1, 2, \ldots, \quad \text{by condition (ii)}. \end{aligned}$$

Hence,

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} n_{(k)}B_{n-k}(0)\frac{x^k}{k!} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}x^k \\ &= \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \end{aligned} \tag{2}$$

[We have replaced the index of summation, k, by $n-k$ in the last sum.]

General formula for the Bernoulli polynomials.

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \tag{3}$$

Remark.. The problem of uniqueness of the Bernoulli polynomials now depends only on whether or not the Bernoulli numbers are uniquely determined. We can establish this by using condition (iii).

For $n \geq 1$ we have

$$\int_0^1 B_n(x)\,dx = 0.$$

And so,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}B_k\frac{x^{n-k+1}}{n-k+1}\Big|_0^1 \\
&\quad=\sum_{k=0}^{n}\binom{n}{k}\frac{B_k}{n-k+1}=\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k \\
&\quad=0,\quad \text{for } n\geq 1. \qquad (4)
\end{aligned}$$

Condition (i) and equation (4) now define a recurrence relation for the Bernoulli numbers, proving that they are uniquely determined.

Recurrence Formula for the Bernoulli numbers.

$$B_0 = 1,$$
$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0, \quad \text{for } n = 1,2,3,\ldots. \qquad (5)$$

Displaying the first few equations given by (5), shows how easy it is to determine the Bernoulli numbers.

$$\begin{aligned}
2B_1+1 &= 0,\\
3B_2+3B_1+1 &= 0,\\
4B_3+6B_2+4B_1+1 &= 0,\\
5B_4+10B_3+10B_2+5B_1+1 &= 0,\\
6B_5+15B_4+20B_3+15B_2+6B_1+1 &= 0,\\
7B_6+21B_5+35B_4+35B_3+21B_2+7B_1+1 &= 0,\\
\cdots \quad &\cdots
\end{aligned}$$

Note. In particular, formula (5) shows that the Bernoulli numbers must all be rational.

Example 2. Some Bernoulli Numbers:

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{65}$

We can also use (**??**) and (5) to write down any number of Bernoulli polynomials.

Some Bernoulli Polynomials

$$
\begin{aligned}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \\
\vdots &\qquad \vdots
\end{aligned}
$$

Some Propoerties of the Bernoulli Numbers

Some simple deductions from the definitions above will increase our understanding of the Bernoulli polynomials and numbers.

Lemma 5.1. $B_n(1) = B_n(0)$, *for* $n \neq 1$.

Proof. The case $n = 0$ is easy. If $n \geq 2$ then by condition (ii),

$$
\begin{aligned}
0 &= \int_0^1 nB_{n-1}(x)\,dx = \int_0^1 B_n'(x)\,dx \\
&= B_n(1) - B_n(0).
\end{aligned}
$$

Hence $B_n(1) - B_n(0) = 0$ for $n \neq 1$. □

Lemma 5.2. $\Delta B_n(x) = B_n(x+1) - B_n(x) = nx^{n-1}$.

Proof. To obtain (**??**) we used the Maclaurin expansion of $B_n(x)$. Let's use the more general Taylor expansion on $B_n(x+1)$.

$$
\begin{aligned}
B_n(x+1) &= \sum_{k\geq 0} B_n^{(k)}(1)\frac{x^k}{k!} \\
&= \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(1)x^k \qquad (5.1) \\
&= \sum_{k=0}^{n} \binom{n}{k} B_k(1)x^{n-k} \qquad (6)
\end{aligned}
$$

From (**??**) and (5) we now have

$$
\begin{aligned}
B_n(x+1) - B_n(x) &= \sum_{k=0}^{n} \binom{n}{k} (B_k(1) - B_k(0))x^{n-k} \\
&= \binom{n}{1}(B_1(1) - B_1(0))x^{n-1} \qquad \text{(By Lemma 1.1)} \\
&= nx^{n-1}.
\end{aligned}
$$

□

Example 3.

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\Big(n^{p+1} + \binom{p+1}{1}B_1 n^p + \binom{p+1}{2}B_2 n^{p-1} + \cdots + \binom{p+1}{p}B_p n\Big).$$

From Chapter 2 §5 we have,

$$\begin{aligned}\sum_{k=0}^{n-1} k^p &= \Delta^{-1}x^p\Big|_0^n = \frac{1}{p+1}B_{p+1}(x)\Big|_0^n \\ &= \frac{1}{p+1}\big(B_{p+1}(n) - B_{p+1}(0)\big) \\ &= \frac{1}{p+1}\Big(\sum_{k=0}^{p+1} B_k n^{p+1-k} - B_{p+1}\Big) \\ &= \frac{1}{p+1}\Big(n^{p+1} + \binom{p+1}{1}B_1 n^p + \binom{p+1}{2}B_2 n^{p-1} + \cdots + \binom{p+1}{p}B_p n\Big).\end{aligned}$$

Note. In symbolic notation (replacing subsscripts by superscripts), we can write this in the easy to remember form,

$$\sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\big((n+B)^{p+1} - B^{p+1}\big).$$

Lemma 5.3. ***(Exponential generating function for the Bernoulli numbers.)***

$$B(t) = \sum_{n\geq 0} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}.$$

Proof. By (5), we can write

$$\sum_{k=0}^{n+1}\binom{n+1}{k}B_k = S_{n+1}, \qquad \text{for } n = 1, 2, \ldots,$$

or

$$B_n = \sum_{k=0}^{n}\binom{n}{k}B_k, \qquad \text{for } n = 2, 3, \ldots.$$

Also

$$\begin{aligned}B_1 + 1 &= \binom{n}{0}B_0 + \binom{1}{1}B_1, \\ B_0 &= \binom{0}{0}B_0.\end{aligned}$$

Hence,

$$\begin{aligned}B(t) &= -t + \sum_{n\geq 0}\Big(\sum_{k=0}^{n}\binom{n}{k}B_k\Big)\frac{t^n}{n!} \\ &= -t + \sum_{n\geq 0}\sum_{k=0}^{n}\Big(\frac{B_k t^k}{k!}\Big)\Big(\frac{t^{n-k}}{(n-k)!}\Big).\end{aligned}$$

The last double sum is just the convolution of two generating functions, $B(t)$ and e^t (see Chapter 2 §2). Hence,

$$B(t) = -tB(t)e^t$$

or

$$B(t)(e^t - 1) = t,$$

from which the result follows. □

Corollary 5.4. *$B_n = 0$ if $n \neq 1$ and n is odd.*

Proof. By Lemma 1.3,

$$\begin{aligned} 1 + \sum_{n\geq 0} B_n \frac{t^n}{n!} &= \frac{t}{e^t - 1} - B_1 t \\ &= \frac{t}{2}\Big(\frac{e^t + 1}{e^t - 1}\Big) \\ &= \frac{t}{2}\Big(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\Big) = \frac{t}{2}\coth\frac{t}{2}. \end{aligned}$$

But $\frac{t}{2}\coth\frac{t}{2}$ is an even function, hence all odd coefficients are 0. □

5.1.1 Exercises with Answers

1. Determine

(i) $\displaystyle\sum_{k=0}^{n} k^2$ [$(n(n+1)/2)^2$]

(ii) $\displaystyle\sum_{k=0}^{n} k^4$ [$n^5/5 + n^4/2 + n^3/3 - n/30$]

2. Suppose condition (iii)* for the standard polynomials is replaced by

(iii)** $P_n(a) = 0$ for $n \geq 1$.

What sequence of polynomials do you get? [$\{(x-a)^n\}$]

3. Find a closed form for the exponential generating function of the sequence, $1, 2B_1, 2^2B_2, 2^3B_3, \ldots$. [$t\coth t - t$]

4. Show that $B_n(1-x) = (-1)^n B_n(x)$.

5.2 The Euler-Maclaurin Summation Formula

The starting point for this useful formula is the observation

$$\int_0^1 f(t)\,dt = \int_0^1 f(t)B_0(t)\,dt. \tag{8}$$

[The few constraints on the function f which are needed will become clear as we proceed. Basically the precision of the formula depends on the number of derivatives that f has.]

Now integrate (8) by parts:

$$\begin{aligned}\int_0^1 f(t)\,dt &= f(t)B_1(t)\Big|_0^1 - \int_0^1 f'(t)B_1(t)\,dt \\ &= f(1)B_1(1) - f(0)B_1(0) - \int_0^1 f'(t)B_1(t)\,dt \\ &= \tfrac{1}{2}(f(0)+f(1)) - \int_0^1 f'(t)B_1(t)\,dt \qquad (9)\end{aligned}$$

If we change the limits of integration in (8) from 0 and 1, to k and $k+1$, then we find

$$\int_k^{k+1} f(t)\,dt = f(k+1)B_1(k+1) - f(k)B_1(k) - \int_k^{k+1} f'(t)B_1(t)\,dt,$$

which is not easily expressible in terms of the Bernoulli number B_1. To get around this difficulty we use, instead of the Bernoulli polynomials $B_n(x)$, the 1-periodic extensions, $\Psi_n(x)$, defined by

$$\begin{aligned} &\Psi_n(x) = B_n(x), && \text{for } 0 \le x < 1, \\ \text{and} \qquad &\Psi_n(x+1) = \Psi_n(x), && \text{for all } x \in \mathbb{R}. \end{aligned}$$

We now replace (8) by,

$$\int_k^{k+1} f(t)\,dt = \int_k^{k+1} f(t)\Psi_0(t)\,dt. \qquad (10)$$

Integrating (10) by parts, now yields

$$\int_k^{k+1} f(t)\,dt = f(t)\Psi_1(t)\Big|_{k+0}^{k+1-0} - \int_k^{k+1} f'(t)\Psi_1(t)\,dt. \qquad (11)$$

Here the last integral is improper since $\Psi_1(x)$ is discontinuous at all integer values of x, and $g(t+0)$ has the usual meaning $g(t+0) = \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} g(t+\epsilon)$.

Now $\Psi_1(k+0) = \Psi_1(0+0) = B_1(0) = -\frac{1}{2}$ and $\Psi_1(k+1-0) = \Psi_1(1-0) = B_1(1) = \frac{1}{2}$. Hence we have the general formula,

$$\int_k^{k+1} f(t)\,dt = \tfrac{1}{2}(f(k)+f(k+1)) - \int_k^{k+1} f'(t)\Psi_1(t)\,dt. \qquad \text{for } k = 0,1,2,3,\ldots. \qquad (12)$$

Now suppose a and b are any integers, with $a \le b$, then summing the equations in (12) from $k=a$ to $k=b-1$ yields

$$\int_a^b f(t)\,dt = \sum_{k=a}^{b} f(k) - \tfrac{1}{2}(f(a)+f(b)) - \int_a^b f'(t)\Psi_1(t)\,dt. \qquad (13)$$

Equation (13) can be rewritten to give us the *first version of the Euler-Maclaurin Summation Formula*,

$$\sum_{k=a}^{b} f(k) = \int_a^b f(t)\,dt + \tfrac{1}{2}(f(a)+f(b)) + \int_a^b f'(t)\Psi_1(t)\,dt. \qquad (14)$$

The General Euler-Maclaurin Summation Formula

We can continue integrating by parts in (12) [until we run out of derivatives for f], using the s'tep-ladder' property of the function $\Psi_m(x)$. In fact the situation is easier for $m > 1$ since the periodic extensions are all continuous, except for $m = 1$, due to Lemma 1.1.

Thus,

$$\begin{aligned}\int_k^{k+1} f(t)\,dt &= \tfrac{1}{2}]((f(k)+f(k+1)) - \Big(\tfrac{1}{2}f'(t)\Psi_2(t)\Big|_k^{k+1} - \frac{1}{2}\int_k^{k+1} f'(t)\Psi_2(t)\,dt\Big) \\ &= \tfrac{1}{2}]((f(k)+f(k+1)) - \tfrac{1}{2}(f'(k+1)\Psi_2(1) - f'(k)\Psi_2(0)) + \frac{1}{2}\int_k^{k+1} f'(t)\Psi_2(t)\,dt \\ &= \tfrac{1}{2}]((f(k)+f(k+1)) - \tfrac{1}{2}B_2((f'(k+1) - f'(k)) + \frac{1}{2}\int_k^{k+1} f'(t)\Psi_2(t)\,dt.\end{aligned}$$

Repeated integration by parts yields,

$$\begin{aligned}\int_k^{k+1} f(t)\,dt = \tfrac{1}{2}]((f(k)+f(k+1)) + \sum_{i=1}^{n}(-1)^{i-1}\frac{B_i}{i!}(f^{(i-1)}(k+1) - f^{(i-1)}(k)) \\ + \frac{(-1)^m}{m!}\int_k^{k+1} f^{(m)}(t)\Psi_m(t)\,dt.\end{aligned} \tag{15}$$

Summing (15) from $k = a$ to $k = b-1$ and rearranging the result gives the *general Euler-Maclaurin summation formula.*

The General Euler-Maclaurin Summation Formula

$$\sum_{k=a}^{b} f(k) = \int_a^b f(t)\,dt + \tfrac{1}{2}(f(a)+f(b)) + \sum_{i=2}^{m}(-1)^i\frac{B_i}{i!}(f^{(i-1)}(b) - f^{(i-1)}(a)) + R_m, \tag{16}$$

$$\text{where } R_m = \frac{(-1)^{m-1}}{m!}\int_a^b f^{(m)}(t)\Psi_m(t)dt.$$

Example 4. (Sum of cubes.) Let $f(x) = x^3$, $a = 1$, $b = n$. Then

$$f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6, \quad f^{(4)}(x) = f^{(5)}(x) = f^{(6)}(x) = \cdots = 0.$$

Hence the Euler-Maclaurin formula gives

$$\begin{aligned}1^3 + 2^3 + 3^3 + \cdots + n^3 &= \int_0^1 x^3\,dx + \tfrac{1}{2}(0^3 + n^3) + \frac{B_2}{2!}(3n^2 - 3\cdot 0^2) \\ &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \Big(\frac{n(n+1)}{2}\Big)^2.\end{aligned}$$

(An identity we have seen before.)

Example 5. (Euler's constant.) Let $f(x) = \frac{1}{x}$, $a = 1$ and $b = n$ in the first version, formula (14).

$$\begin{aligned}1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} &= \int_1^n \frac{dt}{t} + \tfrac{1}{2}(1 + \frac{1}{n}) + \int_1^n f'(t)\Psi_1(t)\,dt \\ &= \log n + \tfrac{1}{2} + \frac{1}{2n} - \int_1^n \frac{\Psi_2(t)}{t^2}\,dt.\end{aligned}$$

This gives us a useful formula for Euler's constant γ, allowing us to calculate it to any number of decimal places.

$$\gamma = \lim_{n\to\infty}\Big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\Big) = \tfrac{1}{2} - \int_1^\infty \frac{\Psi_1(t)}{t^2}\,dt. \tag{17}$$

Note. The harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ grows like $\log n$, both diverging to ∞, but extremely slowly. For exaxmple, 83 terms are required to reach a total of 5 ($\log 83 \approx 4.4$), 12367 terms are need to reach a total of 10 ($\log 12367 \approx 9.4$), and we need 1.5×10^{43} terms to get to a sum of 100 ($\log 1.5\cdot 10^{43} \approx 99.4$).

Note. The first few decimal places of γ are 0.57721566490153286061, however very little is known about γ. For example, we don't even know if it is irrational let alone transcendental.

Example 6. (Stirling Series.) Let $f(x)=\log x$, $a=1$, $b=n$ in formula (16).

$$f'(x)=\frac{1}{x},\quad f''(x)=-\frac{1}{x^2},\quad f^{(3)}(x)=\frac{2}{x^3},\quad \ldots,\quad f^{(k)}(x)=(-1)^{k-1}\frac{(k-1)!}{x^k},\ldots.$$

Hence,

$$\begin{aligned}\sum_{k=1}^n \log k = \int_1^n &\log t\,dt + \tfrac{1}{2}(\log 1+\log n)\\ &+\sum_{k=2}^m \frac{b_k}{k!}(f^{(k-1)}(n)-f^{(k-1)}(1)) + R_m,\end{aligned} \tag{18}$$

where,

$$\begin{aligned}R_m &= \frac{(-1)^{m-1}}{m!}\int_1^n f^{(m)}(t)\Psi_m(t)\,dt = \frac{1}{m!}\int_1^\infty \frac{(m-1)!}{t^m}\Psi_m(t)\,dt\\ &= \frac{1}{m}\int_1^n \frac{\Psi_m(t)}{t^m}\,dt.\end{aligned}$$

Now $\int_1^n \log t\,dt = n\log n-(n-1)$, hence (18) can be written as

$$\begin{aligned}\log n! = n\log n &- (n-1) + \tfrac{1}{2}\log n\\ &+\sum_{k=2}^m (-1)^{k-1}\frac{B_k}{k!}\Big((-1)^{k-2}\frac{(k-2)!}{n^{k-1}} - (-1)^{k-2}(k-2)!\Big) + R_m.\end{aligned} \tag{19}$$

Or, alternatively,

$$\log\Big(\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}\Big) = 1 - \sum_{k=2}^m \frac{B_m}{k(k-1)}\Big(\frac{1}{n^{k-1}}-1\Big) + R_m, \tag{20}$$

where $R_m = \int_1^n \frac{\Psi_m(t)}{t^m}\,dt$.

Remark. We are close to Stirling's formula here, but a complete development is beyond the scope of this book. It can be shown that as n approaches ∞, the right-hand side of (**??**) approaches $\log\sqrt{2\pi}$. This implies that $n!/(\sqrt{2\pi}n^{n+1/2}e^{-n})$ approaches 1 as n approaches ∞, another way of stating Stirling's formula. For a detailed account see the book by R. Graham, D. Knuth and O. Patashnik, "*Concrete Mathematics: A Foundation for Computer Science*" Addison-Wesley (1989) (in particular, pages 467-475).

5.3 Problems 4

1. Use the Euler-Maclaurin summation formula to find $\sum_{k=0}^{n} k^2$; $\sum_{k=0}^{n} k^4$.

2. Show that the non-zero Bernoulli numbers alternate in sign.

3. Show that
$$\gamma = \Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\Big) - \frac{1}{2n} + \frac{1}{12n^2} - R_2,$$
where $\frac{1}{252n^6} \le R_2 \le \frac{1}{120n^4}$.

The next few exercises require some elementary knowledge of Fourier series.

4.

(i) Show that the 1-periodic extension, $\Psi_1(x)$, of the first Bernoulli polynomial $B_1(x)$ has Fourier series
$$-\frac{1}{\pi}(\sin 2\pi x + \frac{1}{2}\sin 4\pi x + \frac{1}{3}\sin 6\pi x + \cdots) = \begin{cases} \Psi_1(x), & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

(ii) Show that
$$\Psi_n(x) = \begin{cases} (-1)^{\frac{n}{2}+1}\frac{2}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\cos 2\pi kx}{k^n}, & \text{if } n \text{ is even and } n > 0, \\ (-1)^{\frac{(n+1)}{2}}\frac{2}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\sin 2\pi kx}{k^n}, & \text{if } n \text{ is odd and } n > 1. \end{cases}$$

(iii) Show that

(a) $$\sum_{k=1}^{\infty}\frac{1}{k^2n} = (-1)^{n-1}\frac{\pi^{2n}2^{2n-1}B_{2n}}{(2n)!}, \qquad \text{for } n = 1, 2, 3, \ldots,$$

(b) $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2n} = (-1)^{n-1}\frac{\pi^{2n}(2^{2n}-1)B_{2n}}{2(2n)!}, \qquad \text{for } n = 1, 2, 3, \ldots,$$

(c) $$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)^2n} = (-1)^{n-1}\frac{\pi^{2n}(2^{2n+1}-1)B_{2n}}{(2n)!}, \qquad \text{for } n = 1, 2, 3, \ldots,$$