4 Pólya Theory

Pólya theory is based on the simple idea that, for example it may be easiest to count a flock of sheep by counting their feet and then dividing by four. We have already seen this principle used in counting circular arrangements with n different letters (we counted the linear arrangements, $n!$, and divided by the number of letters, n). In this chapter we will use it to solve problems like

- counting the number of necklaces with n beads of c possible colours.
- counting the chemical compounds which can be derived by the substitution of a given set of radicals in a given molecular structure.

4.1 Burnside's Lemma

In this section we prove the fundamental lemma on which Pólya theory is based. This lemma gives a formula for the number of orbits of a group G acting on a set Ω. In our previous analogy, the orbits are the sheep, Ω is the set of fett and G determines the relationship between the feet and the sheep.

We will be concerned in this chapter only with finite sets and finite groups.

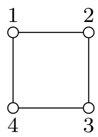
Definition. A group G acts on a finite set Ω if for all $\alpha \in \Omega$ and $g \in G$, $g(\alpha)$ is defined as an element of Ω, and in addition

(i) $gh)(\alpha) = g(h(\alpha)$, for all $g, h \in G$ and $\alpha \in \Omega$.

(ii) $1(\alpha) = \alpha$, for all $\alpha \in \Omega$.

We say that g *moves* α to $g(\alpha)$.

Note. The correspondence $\theta\colon \alpha \mapsto g(\alpha)$, defines a *permutation* of Ω, since if $\theta(\alpha_1) = \theta(\alpha_2)$, then $g(\alpha_1) = g(\alpha_2)$ and so $g^{-1}(g(\alpha_1)) = g^{-1}(g(\alpha_2))$ and thus $1(\alpha_1) = 1(\alpha_2)$, and $\alpha_1 = \alpha_2$ by (ii). This shows that θ is one-to-one and so a permutation of Ω, since Ω is finite.

Example 1. We will describe an action of the group G of symmetries of a square on the set Ω of colourings of the vertices of the square, 

. There are sixtenn ways in which we can colour the vertices with two colours, black (B), and white (W). Each of these is called a *colour scheme*, and the set of colour schemes is the set Ω.

First, we describe the group of symmetries of the square.

There are two types of transformations of the square which preserve its structure; frou rotations, one of which is the identity transformation 1 which leaves the square fixed, and four reflections. These form the dihedral group D_4. THe rotations themselves for a subgroup of D_4, generated by the permutation, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, which represents a clockwise rotation of the square through $90°$. σ^2, σ^3 are clockwise rotations through $180°$ and $270°$ repsectively. Clearly, $\sigma^2 = (13)(24)$, $\sigma^3 = (1432)$, $\sigma^4 = 1$.

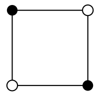
Developed by George Pólya (1935), to count collections of objects possessing some symmetry, for example chemical compounds. See Pólya's "*Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und Chemishe Verbingungen*", Acta Math. **68** (1937), 145-254.

The four reflections can be expressed in terms of σ and the reflection $\tau = (12)(34)$. Thus, interpreting $\sigma\tau$ to mean that τ acts first and σ second, we have

$$(13) = \sigma\tau, \quad (24) = \sigma^3\tau, \quad (14)(23) = \sigma^2\tau.$$

Next we describe the natrual action of the group of symmetris on the set of colour schemes.

The colour schemes can be displaysed as actual squares, e.g.



or, as functions $\phi\colon \{1,2,3,4\} \to \{B, W\}$, e.g.

$$1 \mapsto B, \quad 2 \mapsto W, \quad 3 \mapsto B, \quad 4 \mapsto W.$$

The elements of D_4 act on the colour schemes of the square in the following way. If ϕ is a colour scheme and $g \in D_4$, then

$$(g\phi)(i) = \phi(g^{-1}(i)), \qquad \text{for} \quad i = 1,2,3,4.$$

Let's check that this is an action of D_4 on Ω.

$$(1\phi)(i) = \phi(1(i)) = \phi(i), \quad \text{for } i = 1,2,3,4.$$
$$\text{Hence} \qquad (1\phi) = \phi, \quad \text{so (ii) holds.}$$

If $g, h \in G$, then

$$\begin{aligned} ((gh)\phi)(i) &= \phi((gh)^{-1}(i)) = \phi(h^{-1}(g^{-1}(i))) \\ &= (h\phi)(g^{-1}(i)) \\ &= (g(h\phi))(i). \end{aligned}$$
$$\text{So,} \qquad (gh)(\phi) = g(h(\phi)), \quad \text{for all } \phi \in \Omega. \text{ So (i) holds.}$$

Note. If ϕ is a colour scheme of the square, then geometrically, $\sigma\phi$ would be the colour scheme obtained by rotating ϕ clockwise through $90°$.

Three important sets related to group actions

Definition. The *orbit* of $\alpha \in \Omega$ under the action of the group G is the set of eleents of Ω to which α is moved by elements of G, i.e.,

$$\mathrm{Orb}_G(\alpha) = \{g(\alpha) \mid g \in G\},$$

i.e. the set of elements of Ω which α is moved to by elements of G.

Definition. The *stabilizer* of $\alpha \in \Omega$ under the action of the group G is the set of group elements which fix α, i.e.,

$$G_\alpha = \{g \in G \mid g(\alpha) = \alpha\},$$

i.e. the set of group elements which don't move α.

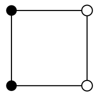
Definition. The set of elements of Ω *fixed* by $g \in G$ is

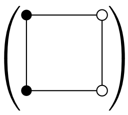
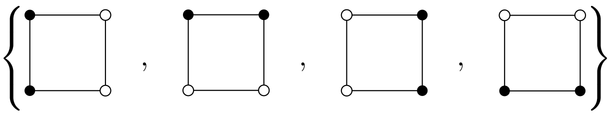
$$\mathrm{Fix}_\Omega(g) = \{\alpha \in \Omega \mid g(\alpha) = \alpha\},$$

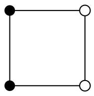
i.e. the set of elements of Ω which are not moved by g.

Note. The sets $\mathrm{Orb}_G(\alpha$ and $\mathrm{Fix}_\Omega(g)$ are subsets of Ω, whereas G_α is a subset of G. In fact G_α is a subgroup of G since,by (ii), $1 \in G_\alpha$ and if $g, h \in G_\alpha$, then clearly $g^{-1}, h^{-1}, gh \in G_\alpha$.

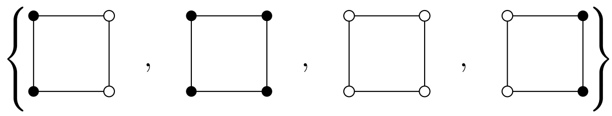
Let's look at these sets in the context of Example 1.

Example 2. The orbit of  is the set

Orb_{D_4}  $=$ .

If $\alpha =$  then $G_\alpha = \{1, (14)(23)\}$.

If $g = (14)(23)$ then

$\mathrm{Fix}_\Omega((14)(23)) =$ 

There is an imporatnt numerical identity which shows that the sizes of the sets $\mathrm{Orb}_G(\alpha$ and G_α are determined by the other, once $|G|$ is known.

Lemma 4.1.
$$|G_\alpha||\mathrm{Orb}_\Omega(\alpha)| = |G|, \quad \textit{for } \alpha \in \Omega.$$

Proof. The idea is to look at the lset of left cosets of G_α, i.e. then sets $xG_\alpha = \{xy \mid y \in G_\alpha\}$.
These all have the same size as G_α, since the mapping,
$$y \mapsto xy,$$
is a one-to-one correspondence between G_α and xG_α.

Moreover two cosets x_1G_α and x_2G_α are either identical or disjoint. To see this suppose $z \in x_1G_\alpha \cap x_2G_\alpha$. Then $z = x_1y_1 = x_2y_2$ for some $y_1, y_2 \in G_\alpha$. Hence $x_1 = x_2y_2y_1^{-1} \in x_2G_\alpha$ and so $x_1G_\alpha \subseteq x_2G_\alpha$. Similarly $x_2G_\alpha \subseteq x_1G_\alpha$. This shows that if two cosets have an element in common then they are identical.

This means that G is partitioned by cosets $x_1G_\alpha, x_2G_\alpha, x_3G_\alpha, \ldots, x_kG_\alpha$, all of the same size.

In other words,
$$|G = k|G_\alpha|.$$
It remains to show that $k = |\mathrm{Orb}_G(\alpha)|$. We do this by showing that
$$\theta\colon x_iG_\alpha \longmapsto x_i(\alpha),$$
defines a one-to-one correspondence between the set of cosets,
$$\{x_1G_\alpha, x_2G_\alpha, \ldots, x_kG_\alpha\},$$
and $\mathrm{Orb}_G(\alpha)$. This will complete the proof of the lemma.

First, $\theta(x_iG_\alpha) = \theta(x_jG_\alpha) \Rightarrow x_i(\alpha) = x_j(\alpha) \Rightarrow x_j^{-1}x_i \in G_\alpha \Rightarrow x_iG_\alpha = x_jG_\alpha$. So θ is one-one. On the other hand, if $g(\alpha) \in \mathrm{Orb}_G(\alpha)$, then $g \in x_iG_\alpha$ for some i. So $g = x_iy$, for some $y \in G_\alpha$, and hence $g(\alpha) = (x_iy)(\alpha) = (x_i(y(\alpha)) = x_i(\alpha)$. Then θ is also onto, hence a one-one correspondence as asserted. ☐

Note. Just as the left cosets of G_α, for given $\alpha \in \Omega$, partition G, it also happens that the orbits of G are all disjoint or equal, and hence partition Ω. To see this, suppose

$$\beta \in \mathrm{Orb}_G(\alpha_1) \cap \mathrm{Orb}_G(\alpha_2).$$

Then $\beta = g(\alpha_1) = h(\alpha_2)$, for some $g, h \in G$. But then $h^{-1}g(\alpha_1) = \alpha_2$, which forces α_2 to be in the same orbit as α_1. So $\mathrm{Orb}_G(\alpha_1) = \mathrm{Orb}_G(\alpha_2)$.

We are ready to prove the main result of this section.

Lemma 4.2. ***(Burnside's Lemma)*** *The number of orbits of the group G acting on the finite set Ω is,*

$$\frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}_G(g)|. \tag{3}$$

Proof. Let $\Omega_1, \Omega_2, \dots, \Omega_n$, be the orbits of G acting on Ω. Then

$$\Omega = \Omega_1 \cup \Omega_1 \cup \cdots \cup \Omega_n.$$

Clearly,

$$\mathrm{Fix}_\Omega(g) = \mathrm{Fix}_{\Omega_1}(g) \cup \cdots \cup \mathrm{Fix}_{\Omega_n}(g),$$

so,

$$\sum_{g\in G} |\mathrm{Fix}_\Omega(g)| = \sum_{g\in G}\sum_{i=1}^{n} |\mathrm{Fix}_{\Omega_i}(g)|,$$

counts the number of eleemts in this set of points,

$$\begin{aligned} X &= \{(\alpha, g) \mid \alpha\imath\Omega, g \in G, g(\alpha) = \alpha\} \\ &= \{(\alpha, g) \mid \alpha\imath\Omega, g \in G, g \in \mathrm{Fix}_\Omega(g)\} \\ &= \{(\alpha, g) \mid \alpha\imath\Omega, g \in G, g \in G_\alpha\}. \end{aligned}$$

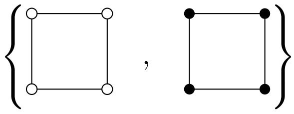
Hence,

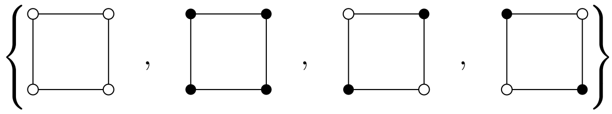
$$\begin{aligned} \sum_{g\in G} |\mathrm{Fix}_\Omega(g)| &= \sum_{i=1}^{n}\sum_{\alpha\in\Omega_i} |G_\alpha| \\ &= \sum_{i=1}^{n}\sum_{\alpha\in\Omega_i} \frac{|G|}{|\mathrm{Orb}_G(\alpha)|}, \quad \text{by Lemma 1,} \\ &= \sum_{i=1}^{n}\sum_{\alpha\in\Omega_i} \frac{|G|}{|\Omega_i|} \\ &= \sum_{i=1}^{n} |\Omega_i| \frac{|G|}{|\Omega_i|} = n|G|. \end{aligned}$$

Thus $n|G| = \sum_{g\in G} |\mathrm{Fix}_\Omega(g)|$, which completes the proof. □

Example 3. Consider the group $G = \langle\sigma\rangle$ (the rotations of Example 1), generated by $\sigma = (1234)$, and let Ω be the set of colour schemes of the square by two colours. We find

$$\mathrm{Fix}_\Omega(1) = \Omega,$$

$$\mathrm{Fix}_\Omega(\sigma) = \mathrm{Fix}_\Omega(\sigma^2) =$$


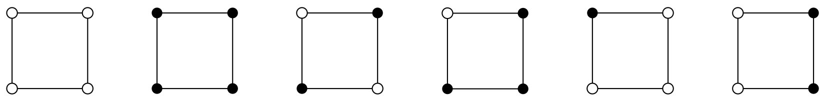
$$\mathrm{Fix}_\Omega(\sigma^3) =$$


Hence by Burnside's Lemma, there number of colour schemes of the square, distinct up to rotation, is

$$\tfrac{1}{4}(16 + 2 + 2 + 4) = 6.$$

(Another interpretation, : the number of colour schemes of the square, distinct circular, planar, arrangements of length four which can be made with beads of two colours, is 6.)

We can easily list the possible distinct colourings:



4.1.1 Exercises with Answers

1. Let $G = D_4$, and let Ω be the set of colour schemes of the square in Example 1. Find the number of orbits of G acting on Ω, and interpret this as a problem of counting necklaces. [6]

2. Show that the set of rotations and reflections of the square form a group.
[Hint: Write out the multiplication table.]

3.

(i) Determine the number of distinct ways of colouring the vertices of an equilateral triangle, using two colours, if the triangle is free to move in the plane, but not in 3-space. If the triangle is free to move in 3-space. [4; 4]

(ii) Do part (i) including a third colour. [11;10]

4.2 The Cycle Index Polynomial

In Example 1 we began with a group G of permutations of the set $\{1, 2, 3, 4\}$ (the rotations and reflections of a square). We found that G acted on a larger set Ω of colour schemes of the square (that G was also a permutation group on the set Ω). So far we haven't taken much notice of the actual elements of G.

A great insight of Pólya was to observe that elements of G with the same *cycle structure* made the same contribution to the sets of fixed points. He invented the *cycle index polynomial* to keep track of the cycle structure of the elements of G.

Definition. Let G be a permutation group on the set X, where $|X| = n$. For $g \in G$, let $b_k(g)$ be the number of cycles of g of length k. Then the *cycle index polynomial* of G, as a permutation group on the set X, is the polynomial in n variables $x_1, x_2, \ldots, x_n$,

$$P_{(G,X)}(x_1, x_2, \ldots, x_n) = \frac{1}{|G|} \sum_{g\in G} x_1^{b_1(g)} x_2^{b_2(g)} \cdots x_n^{b_n(g)}.$$

If X is implicit from the context, we may write $P_{(G,X)} = P_G$.

Application to counting problems

Let C be a finite set (it's good to think of C as a set of colours). Let Ω be the set of all functions $\phi\colon X \to C$ (we can think of Ω as the set of all possible colour sequences of the points in X).

The group G acts on Ω in the following natural way. For $g \in G$ and $\phi \in \Omega$, we define,

$$(g\phi(x) = \phi(g^{-1}x), \qquad \text{for all } x \in X.$$

[The verification that this is a group action is the same as that given in Example 1.]

Terminology. As before, an element of Ω is often called a *colour scheme*, and an orbit of G acting on Ω is a *pattern*.

Theorem 4.3. ***(Pólya)*** *The number of patterns of G acting on Ω is*

$$P_{(G,X)}(|C|, |C|, \ldots, |C|).$$

Proof. By Burnside's Lemma, the number of patterns (orbits) is

$$\frac{1}{|G|} \sum_{g\in G} |\mathrm{Fix}_\Omega(g)|,$$

where $\mathrm{Fix}_\Omega(g) = \{\phi \in \Omega \mid g\phi = \phi\}$.
We will see that g fixes a colour scheme ϕ if and only if ϕ colours the elements of each cycle of g with the same colour.
If $g\phi = \phi$, then $(g\phi)(x) = \phi(x)$, so $\phi(g^{-1}x) = \phi(x)$, for all $x \in X$. In particular, this holds for $x = y, g(y), g^2(y), \ldots,$, where $y \in X$.
Hence $\phi(y) = \phi(g^{-1}y) = \phi(g^{-1}g(y)) = \phi(g(y))$, so

$$\phi(y) = \phi(g(y)) = \phi(g^2(y)) = \phi(g^3(y)) = \cdots.$$

THis means that if g fixes ϕ then ϕ assigns the same colour to each element of any cycle of g.
Conversely, if ϕ is such that each cycle of g is coloured with the same colour then $g^{-1}x$ and x have the same colour for each $x \in X$, i.e. $\phi(g^{-1}x) = \phi(x)$, and so $(g\phi)(x) = \phi(x)$, for each $x \in X$. We conclude that $g\phi = \phi$ (g fixes ϕ).
This completes the proof of our assertion, that the elements in $\mathrm{Fix}_\Omega(g)$ are precisely those with colour the elements of any cycle with just one colour.
Hence, $|\mathrm{Fix}_\Omega(g)| = |C|^{b_1(g)}|C|^{b_2(g)} \cdots |C|^{b_n(g)}$, which completes the proof of the theorem. □

Example 4. Let $G = \langle\sigma\rangle$ and Ω be as in Example 3.
The cycle index polynomial of G is

$$P_G(x_2, x_2, x_3, x_4) = \tfrac{1}{4}(x_1^4 + x_2^2 + 2x_4).$$

[In this case, $X = \{1, 2, 3, 4\}$.]
Hence by Pólya's theorem, the number of orbits of G on Ω, $(C = \{B, W\})$ is $\frac{1}{4}(2^4 + 2^2 + 2 \cdot 2) = 6$ as before.

Example 5. How many distinct necklaces can be made containing six beads, if the beads can be chosen from 3 different colours? [Two necklaces will be regarded as equivalent if the beads of one can be brought to the same position by rotations and reflections, as the beads of the other.]

Let G be the group of rotations and reflections of a regular hexagon, and Ω the set containing the 3^6 possible colour schemes of the hexagon, using three colours. Clearly, the answer to our problem is the number of patterns obtained by the action of G on Ω.

The group G is the dihedral group D_4, consisting of six rotations (including the identity) of the hexagon, and six reflections (three about the lines through opposite pairs of vertices, and three about lines through the midpoints of opposite pairs of edges). If we label the vertices $1, 2, 3, 4, 5, 6$, the rotations can be written as

$$1, \qquad \sigma = (123456), \qquad \sigma^2 = (135)(246), \qquad \sigma^3 = (14)(25)(36),$$
$$\sigma^4 = (153)(264), \qquad \sigma^5 = (165432),$$

and the reflections can be written as,

$$(26)(35), \qquad (13)(46), \qquad (15)(24), \qquad (16)(25)(34), \qquad (12)(36)(45), \qquad (14)(23)(56).$$

The cycle index polynomial is,

$$P_G(x_1, x_2, x_3, x_4, x_5, x_6) = \tfrac{1}{12}(x_1^6 + 3x_1^2x_2^2 + 4x_2^3 + 2x + 3^2 + 2x_6).$$

So, by Pólya's theorem, the number of patterns (and hence, distinct necklaces) is

$$\tfrac{1}{12}(3^6 + 3 \cdot 3^2 \cdot 3^2 + 4 \cdot 3^3 + 2 \cdot 3^2 + 2 \cdot 3) = 92.$$

4.2.1 Exercises with Answers

1. Let $G = \langle \sigma \rangle$, where $\sigma = (12345)$ represents the five rotations in the plane of a regular pentagon. Let Ω be the 3^5 colour schemes of the pentagon using three colours. Find the cycle index poylnomial of G and the number of patterns of G acting on Ω. [$\frac{1}{5}(x_1^5 + 4x_5)$; 51]

2. Let V be the set of vertices of a cube and let G be the group of permutations of V produced by rotations of the cube. Determine the cycle index polynomial of G and the number of distinct ways of colouring the vertices of a cube with two colours [distinct with respect to rotations of the cube).
[$\frac{1}{24}(x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2$; 23]

3. Let E be the set of edges of a cube and let G be the group of permutations of E produced by rotations of the cube. Determine the cycle index polynomial of G and the number of distinct ways of colouring the edges of a cube with two colours [distinct with respect to rotations of the cube).
[$\frac{1}{24}(x_1^{12} + 3x_2^6 + 8x_3^4 + 6x_4^3 + 6x_1^2x_2^5$; 218]

4.3 Pólya's Inventory Theorem

In Example 5, we found the number of necklaces of six beads in three colours. In this section, we refine our methods to enable us to count, for example, the number of necklaces of six beads, two of which are red, three green and one blue.

The idea is to introduce a *weight* to each colour scheme. By adjusting the weights we will be able to pick out specific schemes.

Again, let G be a permutation group on the set X, C a set of colours, Ω the set of colour schemes $\phi\colon X \to C$.

Definition. Let A be a commutative ring with identity (the elements of A are called *weights*). Let

$$\omega\colon C \to A$$

be any function with domain C and target A. Then the *weight of a colour scheme* $\phi\colon X \to C$*, with respect to* ω, is,

$$\omega(\phi) = \prod_{x\in X} \omega(\phi(x)).$$

Note. Colour schemes in the same orbit under the action of G have the same weight. To see this consider $g\phi$ for $g \in G$, $\phi \in \Omega$.

$$\begin{aligned}\omega(g\phi) &= \prod_{x\in X} \omega((g\phi)(x)) \\ &= \prod_{x\in X} \omega((\phi)(g^{-1}x)).\end{aligned}$$

As x ranges over X, so does $g^{-1}x$, since g^{-1} is a permutation of X. Hence $\prod_{x\in X} \omega(\phi(g^{-1}x))$ and $\prod_{x\in X} \omega(\phi(x))$ are products of the same elements of A, though possibly in different orders. Since A is commutative, we conclude that $\omega(g\phi) = \omega(\phi)$.

Definition. Suppose Δ is an orbit of G on Ω (a pattern). THe *weight* of Δ is just the weight of any colour scheme in Δ. That is,

$$\omega(\Delta) = \omega(\phi), \qquad \text{for any } \phi \in \Delta.$$

Definition. The *pattern inve3ntory*, I, of the action of G on Ω, with respect to ω, is the sum of the weights of the orbits. That is

$$I = \sum_{\substack{\Delta \text{ an orbit} \\ \text{of } G \text{ on } \Omega}} \omega(\Delta). \tag{3}$$

Our aim is to find an expression for I in terms of the cycle index polynomial. Towards this end we first generalize Burnside's Lemma.

By Lemma 1.1,

$$|\Delta||G_\alpha| = |G|, \qquad \text{for } \phi \in \Delta.$$

So

$$\begin{aligned}\omega(\Delta) &= \frac{1}{|\Delta|}\sum_{\alpha\in\Delta} \omega(\phi) \\ &= \sum_{\alpha\in\Delta} \frac{|G_\phi|}{|G|}\omega(\phi) \\ &= \frac{1}{|G|}\sum_{\alpha\in\Delta} |G_\phi|\omega(\phi).\end{aligned}$$

Hence, (3) can be written as,

$$\begin{aligned} I = \sum_{\substack{\Delta \text{ an orbit} \\ \text{of } G \text{ on } \Omega}} \omega(\Delta) &= \sum_{\substack{\Delta \text{ an orbit} \\ \text{of } G \text{ on } \Omega}} \frac{1}{|G|} \sum_{\alpha\in\Delta} |G_\phi|\omega(\phi) \\ &= \frac{1}{|G|} \sum_{\phi\in\Omega} |G_\phi|\omega(\phi) \\ &= \frac{1}{|G|} \sum_{\phi\in\Omega} \sum_{\substack{g\in G \\ g\phi=\phi}} \omega(\phi) \\ &= \frac{1}{|G|} \sum_{g\in G} \sum_{\substack{\phi\in\Omega \\ g\phi=\phi}} \omega(\phi), \qquad \text{(interchanging sums).} \end{aligned} \tag{4}$$

The identity given in (4) is the desired generalization of Burnside's lemma.

Lemma 4.4. ***(Weighted Burnside Lemma.)***

$$I = \sum_{\substack{\Delta \text{ an orbit} \\ \text{of } G \text{ on } \Omega}} \omega(\Delta) = \frac{1}{|G|} \sum_{g\in G} \sum_{\substack{\phi\in\Omega \\ g\phi=\phi}} \omega(\phi), \tag{5}$$

To see that this is a generalization, take $\omega\colon C \to A$ to be the function defined by $\omega\colon c \mapsto 1$, for $c \in C$. Then $\omega(\Delta) = 1$ for any orbit Δ, so $\displaystyle\sum_{\substack{\Delta \text{ an orbit} \\ \text{of } G \text{ on } \Omega}} \omega(\Delta)$ is the number of orbits of G acting on Ω. Also $\omega(\phi) = 1$ for any $\phi \in \Omega$, so the right-hand side of (5), becomes $\displaystyle\frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}_\Omega(g)|$.

We are ready to prove the main result of this section.

Theorem 4.5. ***(Pólya Inventory Theorem.)***

$$I = \sum_{\substack{\Delta \text{ an orbit} \\ \text{of } G \text{ on } \Omega}} \omega(\Delta) = P_{(G,X)}(p_1, p_2, \ldots, p_n),$$

where $|X| = n$ *and* $p_k = \sum_{c\in C} \omega(c)^k$ *is the* k*th* power sum *of the weights of the colours.*

Proof. From the weighted Burnside Lemma, it remains to prove that

$$\sum_{\substack{\phi\in\Omega \\ g\phi=\phi}} \omega(\phi) = \sum_{\phi\in\mathrm{Fix}_\Omega(g)} \omega(\phi) p_1^{b_1(g)} p_2^{b_2(g)} \cdots p_n^{b_n(g)},$$

where $b_k(g)$ is the number of ccles of g of length k.

We have already seen in Theorem 2.1, that $\mathrm{Fix}_\Omega(g)$ consists precisely of those colour schemes which colour each cycle of G with just one colour. What is the weight of such a colour scheme? To answer this, suppose g consists of t cycles, whose elements define the sets $X_1, X_2, \ldots, X_t$ (which partition X). If $\phi \in \mathrm{Fix}_\Omega(g)$ then,

$$\begin{aligned} \omega(\phi) &= \prod_{x\in X} \omega(\phi(x)) \\ &= \prod_{i=1}^{t} \omega(\phi(x_i))^{|X_i|}, \end{aligned}$$

where x_i is any element of X_i (these are all coloured by the same $\phi \in \mathrm{Fix}_\Omega(g)$).
Hence

$$\begin{aligned}\sum_{\phi\in\mathrm{Fix}_\Omega(g)} \omega(\phi) &= \sum\sum\cdots \sum_{\substack{c_1,\ldots,c_t\\ c_i\in C}} \prod_{i=1}^t \omega(\phi(x_i))^{|X_i|} \\ &= \Big(\sum_{c_1\in C}\prod_{i=1}^t \omega(\phi(x_i))^{|X_1|}\Big)\Big(\sum_{c_1\in C}\prod_{i=1}^t \omega(\phi(x_i))^{|X_1|}\Big)\cdots\Big(\sum_{c_1\in C}\prod_{i=1}^t \omega(\phi(x_i))^{|X_1|}\Big) \qquad (6)\end{aligned}$$

But g has $b_k(g)$ cycles of length k, so (6) can be written as

$$\begin{aligned}\sum_{\phi\in\mathrm{Fix}_\Omega(g)} \omega(\phi) &= \prod_{k=1}^n \Big(\sum_{c\in C}\omega(c)^{|kb_k(g)}\Big) \\ &= \prod_{k=1}^n p_k^{b_k(g)}.\end{aligned}$$

This completes the proof. □

Example 6. Return to Exmaple 5. Suppose the colours are black (B), white (W) and red (R). How many necklaces have at least one red bead? How many have three red beads, two black beads and one white bead?

As before G is the dihedral group D_4, which is a group of permutations on the six vertices of the hexagon. The cycle index polynomial, from Example 5, is,

$$P_G = \tfrac{1}{12}(x_1^6 + 4x_2^3 + 2x_3^2 + 2x_6 + 3x_1^2x_2^2).$$

Let $A = \mathbb{Q}[x]$ and $\omega\colon C \to \mathbb{Q}[x]$ be defined by $\omega(R) = x$, $\omega(B) = \omega(W) = 1$. Then $p_k = x^k + 1^k + 1^k = x^k + 2$, for $k = 1, 2, 3, 4, 5, 6$. So the pattern inventory is, by Theorem 3.2,

$$\begin{aligned}I &= \tfrac{1}{12}\big((x+2)^2 + 4(x^2+2)^3 + 2(x^3+2)^2 + 2(x^6+2) + 3(x+2)^2(x^2+2)^2\big) \\ &= x^6 + 2x^5 + 9x^4 + 16x^3 + 29x^2 + 20x + 15.\end{aligned}$$

The coefficient of x^i now gives the number of necklaces having i red beads [1 has six red beads, 2 have five red beads, . . ., 15 have no red beads.]

The total number of necklaces is,

$$1 + 2 + 9 + 16 + 29 + 20 + 15 = 92,$$

as in Example 5. So the number of necklaces with at least one red bead is $92 - 15 = 77$.

To count the number of necklaces with three red beads, two black beads and one white bead, let $A = \mathbb{Q}[x, y, z]$ and $\omega\colon C \to A$ be defined by $\omega\colon R \mapsto x$, $\omega\colon B \mapsto y$, $\omega\colon W \mapsto z$.

Then the pattern inventory is,

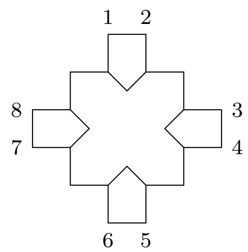
$$\begin{aligned}I = \tfrac{1}{12}\big((x+y+z)^6 &+ 4(x^2+y^2+z^2)^3 + 2(x^3+y^3+z^3)^2 \\ &+ 2(x^6+y^6+z^6) + 3(x+y+z)^2(x^2+y^2+z^2)^2\big).\end{aligned}$$

We want the coefficient of x^3y^2z. The terms involving x^3y^2 are given by

$$\begin{aligned}\tfrac{1}{12}\big(\tbinom{6}{3,2,1}x^3y^2z + 3(2xz)(2x^2y^2)\big) &= \tfrac{1}{12}\big(\frac{6!}{12} + 12\big)\\ &= 6.\end{aligned}$$

So there are six necklaces of length 6, with 3 red beads, 2 black beads and 1 white bead.

Example 7. Porphyrins are chemical compounds derived from porphin which has the structure shown below (somewhat simplified).



The numbers 1 to 8 indicate sites at which substitutions by radicals may take place.

How many porphyrins are there, having two types of radical, four of each kind?

To answer this, we need to determine first the group G, of symmetries of the above structure. This consists of eight elements,

(1) the identity transformation, 1;

(2) three rotations, clockwise by $\pi/2$, π, and $3\pi/2$, giving the elements,

$$\sigma = (1357)(2468), \qquad \sigma^2 = (15)(26)(37)(48), \quad \sigma^3 = (1753)(2864);$$

(3) four reflections: two about the vertical and horizontal axes of the structure respectively, and two about diagonal and off-diagonal lines. These are given by

$$\begin{aligned} \tau &= (12)(38)(47)(56), & \tau\sigma &= (18)(27)(36)(45),\\ \tau\sigma^2 &= (16)(25)(34)(78), & \tau\sigma^3 &= (14)(23)(58)(67). \end{aligned}$$

Hence the cycle index polynomial of G is

$$P_G = \tfrac{1}{8}(x_1^8 + 5x_2^4 + 2x_4^2).$$

If we take $C = \{r, s\}$, $a = \mathbb{Q}[r, s]$ and $\omega\colon C \to A$, defined by $\omega\colon r \mapsto r$, $\omega\colon s \mapsto s$, the pattern inventory is

$$I = \tfrac{1}{8}\big((r+s)^8 + 5(r^2+s^2)^4 + 2(r^4+s^4)^2\big).$$

We want the number of porphyrins with four of each of the radical r and s, which is given by the coefficient of r^4s^4, i.e. by,

$$\tfrac{1}{8}\big(\tbinom{8}{4} + 5\tbinom{4}{2} + 2\cdot 2\big) = 13.$$

Hence there are 13 possible such porphyrins.

See G. Pólya and R.C. Read, *Combinatorial Enumeration of Groups, Graphs and Chemical Compounds*, Springer (1987), p. 124.

4.3.1 Exercises with Answers

1. In Example 7, determine the number of porphyrins having three types of radical, four of the first kind, two of the second and two of the third. [60]

2. A roulette wheel has 15 locations, at regular intervals, to be coloured with three colours. Two colour schemes are considered to be the same if one can be obtained from the other by a rotation in the plane. Determine the cycle index polynomial for the relevant group of rotations, and find the total number of possible patterns. How many patterns have 9 positions coloured with one colour, 5 with the second and 1 position coloured with the third colour? [$\frac{1}{15}(x_1^{15} + 2x_3^5 + 4x_5^3 + 8x_{15})$; 956,635; 2002]

4.4 Problems 3

1. Let $X = \{1, 2, 3\}$ and $G = S_3$ be the group of all permutations of X. Let $C = \{1, 2\}$ and define $\omega\colon C \to \mathbb{Z}[x, y]$ by $\omega(1) = x$ and $\omega(2) = y$.

(i) Write out all 8 functions from X to C and find their weights.

(ii) Find the orbits of G on the functions found in (i).

(iii) Compare your solution to (ii) with the pattern inventory found by Pólya's theorem.

2. (de Bruijn) Let m be a fixed integer, and let $n = 2m + 1$. Consider the 10^n numbers of n digits (a number might start with one or more zeros), each number being printed on a card. Two cards are considered the same if one of them can be transformed into the other by putting it upside down (for example, 0698161 is regarded the same as 1918690, since 0,6,9,8,1 cannot be distinguished from 0,9,6,8,1, respectively, under the transformation). Show that the number of different cards is

$$10^n - \tfrac{1}{2}5^n + \tfrac{3}{2}5^m.$$

3. A cylinder (with circular base) is coloured with n horizontal strips in c colours. In how many ways can this be done?

4. A cylinder (with circular base) is coloured with n vertical strips in c colours. In how many ways can this be done?

5. Write out the cycle index polynomial of the group D_n. Consider the set of all necklaces that can be formed using eight beads coloured red, green or blue. Find the pattern inventory, and determine how many patterns there are with three red beads, two green and three blue beads.

6. Let F be the set of faces of a cube and let G be the group of permutations of F produced by rotations of the cube. Determine teh cycle index polynomial of G and the number of distinct ways of colouring the faces of a cube with c colours (distinct with respect to rotations of the cube).

7. (Bill Unger) Suppose G acts on X and $b_k(g)$ denotes the number of k-cycles in X (of the action of $g \in G$ on X). Also suppose we know $b_1\colon G \to \mathbb{N}$. Show that $b_n\colon G \to \mathbb{N}$ is given by

$$b_n(g) = \frac{1}{n} \sum_{k|n} \mu\big(\tfrac{n}{k}\big) b_1(g^k).$$

Find the cycle index polynomial of G.