1.10 Tutorial 8 MAST30005 Semester I Year 2024: Polynomials and field extensions

1. Show that the following polynomials are irreducible in $\mathbb{Q}[x]$:
$$x^2-12,\qquad 8x^3+4399x^2-9x+2,\qquad \text{and}\qquad 2x^{10}-25x^3+10x^2-30.$$
2. List all monic polynomials of degree ≤ 2 in $\mathbb{F}_3[x]$. Determine which of these are irreducible.
3. Let $f(x)=x^3-5$. Show that $f(x)$ does not factor into three linear polynomials with coefficients in $\mathbb{Q}[\sqrt[3]{5}]$.
4. (a) Find a degree four polynomial $f(x)$ in $\mathbb{Q}[x]$ which has $\sqrt{2}+\sqrt{3}$ as a root.
   (b) Find the degree of the field extension $\mathbb{Q}[\sqrt{2}+\sqrt{3}]$ of $\mathbb{Q}$. (Possible Hint: Any factor of $f(x)$ in $\mathbb{Q}[x]$ is also a factor of $f(x)$ in $\mathbb{C}[x]$, and we can list all these factors)
5. Show that a finite field has order a power of a prime.
6. Show that there are infinitely many irreducible polynomials of any given positive degree in $\mathbb{Q}[x]$.
7. Let F be a field of characteristic p and let q be a power of p. Show that
$$X=\{x\in F\mid x^q=x\}\qquad \text{is a subfield of } F.$$
8. Let α be a complex root of the irreducible polynomial x^3-x+4. Find the inverse of $\alpha^2+\alpha+1$ in $\mathbb{Q}[\alpha]$ explicitly, in the form $a+b\alpha+c\alpha^2$, with $a,b,c\in\mathbb{Q}$.
9. Let F be a field, and α an element that generates a field extension of F of degree 5. Prove that α^2 generates the same extension.
10. Let a be a root of the polynomial x^3-x+1. Determine the minimal polynomial for a^2+1 over $\mathbb{Q}$.
11. (a) Let $a,b,c,d\in\mathbb{C}$ with $ad-bc\neq 0$. Prove that there exists an automorphism σ of $\mathbb{C}(z)$ with $\sigma(z)=\frac{az+b}{cz+d}$ (these are called Mobius transformations)
    (b) Determine the relationship between composition of Mobius transformations and matrix multiplication.
    (c) Show that the automorphisms $\sigma(t)=it$ and $\tau(t)=t^{-1}$ of $\mathbb{C}(t)$ generate a group G that is isomorphic to the dihedral group D_4.
    (d) Let $u=t^4+t^{-4}$. Show that u is fixed under H.
    (e) What is $[\mathbb{C}(t):\mathbb{C}(u)]$?
12. Let F be a field and let $a_1,a_2,\ldots,a_n$ be the roots of a polynomial $f\in F[x]$ of degree n. Prove that $[F[a_1,\ldots,a_n]:F]\le n!$.
13. Let R be an integral domain that contains a field F as a subring and is finite dimensional when viewed as a vector space over F. Prove that R is a field.