MAST30005 ALGEBRA
SEMESTER 1, 2024
PRACTICE CLASS 6 NEW

A *ring* in this tutorial will always mean a commutative unital ring.

Posets

Definition. Let $(A, \leq)$ be a poset. By abuse of privilege of laziness we shall use A for both the poset and the set. We say A is *totally ordered* if for all $a, b \in A$, we have $a \leq b$ or $b \leq a$. We shall say A is *well-ordered* if any nonempty subset $U \subseteq A$ has a least element (an element x of a subset $V \subseteq A$ is *least* if $x \leq y$ for all $y \in V$).

(1) Give an example (if possible) of a poset that is
   (a) Not totally ordered.
   (b) Well ordered.
   (c) Well ordered but not totally ordered.

(2) Prove any set can be well-ordered. More precisely, let A be a set. Show there is a partial order $\leq$ on A such that $(A, \subseteq)$ is well-ordered.

(3) Give an example of a subset of $\mathbb{Q}$ that has no supremum.

(4) Give an example of a subset of $\mathbb{R}$ that has no supremum.

Definition. Let A be a ring, and let I, J be ideals of A. We define the *product* of I and J, denoted IJ to be the set

$$IJ := \left\{ \sum_{i=1}^{r} x_i y_i \mid x_i \in I, y_i \in J \right\},$$

or in english, everything that can be written as a finite sum of something in I times something in J.

(1) Prove IJ is an ideal.

(2) Let $I = 3\mathbb{Z}$ and $J = 15\mathbb{Z}$. Find IJ.

(3) Suppose I and J are both principal. What can you say about IJ?

(4) Show that $IJ \subseteq I \cap J$. Give a counterexample to the converse. Can you give a necessary and sufficient condition for equality to occur in $\mathbb{Z}$?

Finiteness Conditions

Recall that a module M over a ring A is *noetherian* (resp. *artinian*) if it satisfies ACC (resp. DCC).

(5) Let A be a Noetherian ring and let M be a finitely generated module over A. Show that M is also noetherian (Hint: prove it first for M free).

(6) Let A be a ring and suppose every finitely generated A-module is noetherian. Show that A is a noetherian ring.

(7) If we replace noetherian with artinian in the above two questions, does it work?

Principal Ideals

Notation: if A is a ring and $f_1, ..., f_r$ are elements of A, we will abuse our privilege to laziness by writing $A/(f_1, ..., f_r)$ to mean $A/(f_1, ..., f_r)A$.

(8) Show $\mathbb{Z}[\sqrt{-5}]$ is not a PID directly by producing an ideal that's not principal.

(9) Do the same with $\mathbb{Z}[\sqrt{5}]$ and with $\mathbb{C}[x, y]/(y^2 - x^3)$.

(10) As we showed in class, we already know $\mathbb{Z}[\sqrt{-5}]$ is not a PID since 6 has two factorisations into irreducibles. However, write the ideal $6\mathbb{Z}[\sqrt{-5}]$ as the product of two prime ideals. Do you think this factorisation (into prime ideals) unique?

(11) Try write $x\mathbb{C}[x, y]/(y^2 - x^3)$ as a product of prime ideals. Do the same with $(1 + \sqrt{5})\mathbb{Z}[\sqrt{5}]$.

(12) Let $A = \mathbb{C}[x, y]/(y^2 - x^3 + x)$. Show that A is not a PID by finding an ideal that's not principal. However, factor taht ideal into primes.

(13) Let $A = \mathbb{C}[x, y]/(y^2 - x^3 + x)$ again. We will show that A is also not a UFD as follows:
   (a) Consider $\mathbb{C}[x]$, which is also a subring of A. Show there is an automorphism $\sigma : A \to A$ that leaves $\mathbb{C}[x]$ fixed but sends y to $-y$.
   (b) We define a norm $N : A \to \mathbb{C}[x]$ by setting $N(a) = a\sigma(a)$. Show that $N(a)$ actually ends up in $\mathbb{C}[x]$, and use it to find the units (invertible elements) in A.
   (c) Imitate the homework problem in Lecture 18 to show that x, y are irreducible in A. Use this to show that A is not a UFD.

(14) Give an example of a PID with one maximal ideal. How about two? How about seventeen?

Miscellaneous Rings/Modules questions

Here are some random ring questions that I'm not sure which tutorial to put in.

(1) Let R be a ring and let M be an R-module. Suppose that U and V are two submodules of M. Show that

$$M \cong U \oplus V \qquad \text{if and only if} \qquad U \cap V = \{0\} \text{ and } U + V = M.$$

(Hint: it's not true. Find a counterexample).

(2) Let A be a ring. For any ring R, we will let $\mathrm{Hom}(R, A)$ denote the set of ring homomorphisms from R to A. Calculate the following:
   (a) $\mathrm{Hom}(\mathbb{Z}, A)$.
   (b) $\mathrm{Hom}(\mathbb{Z}[x], A)$.
   (c) $\mathrm{Hom}(\mathbb{Z}[x]/(x^2 - 1), A)$.
   (d) $\mathrm{Hom}(\mathbb{Z}[x, y], A)$.
   (e) $\mathrm{Hom}(\mathbb{Z}[x, y]/(xy - 1), A)$.
   (f) $\mathrm{Hom}(\mathbb{Z}[a, b, c, d, e]/(e(ad - bc) - 1), A)$.

(3) Let p be a prime number and let $A \in \mathrm{GL}_{p-2}(\mathbb{Q})$ satisfy $A^p = 1$. Show that $A = 1$.