3.10 Tutorial 4: NEW MAST30005 Semester 1: Proof machine practice

1. The correspondence theorem. Let R be a ring, M an R-module and N an R-submodule of M.

(a) Carefully define the lattice $\mathcal{S}_N^M$ of submodules between N and M.

(b) Carefully state the correspondence theorem.

(c) Carefully prove the correspondence theorem.

2. Tor. Let R be a ring, M an R-module and N an R-submodule of M.
Let R be a integral domain and let M be an R-module.

- The **torsion submodule** of M is

$$\mathrm{Tor}(M) = \{m \in M \mid \text{there exists } a \in R \text{ with } a \neq 0 \text{ and } am = 0\}.$$

- The module M is **free of finite rank** if there exists $r \in \mathbb{Z}_{>0}$ such that $M \cong \mathbb{A}^{\oplus r}$.

Let R be an integral domain and let M be an R-module.

(a) Show that if M is an R-module then $\mathrm{Tor}(M)$ is an R-submodule of M.

(b) Show that if M and N are R-modules then $\mathrm{Tor}(M \oplus N) = \mathrm{Tor}(M) \oplus \mathrm{Tor}(N)$.

(c) Show that $\mathrm{Tor}(R) = 0$.

(d) Show that if $d \in R$ and $d \neq 0$ then $\mathrm{Tor}(R/dR) = R/dR$.

(e) Carefully prove the following proposition.

Proposition 3.22. *Let $\mathbb{A}$ be a PID. Assume that M is an $\mathbb{A}$-module and there exist $r, k \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_k \in (\mathbb{A} - \{0, 1\})/\mathbb{A}^\times$ such that*

$$M \cong \mathbb{A}^{\oplus r} \oplus \Big(\frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}}\Big). \qquad \textit{Then} \qquad \mathrm{Tor}(M) \cong \frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}}.$$

3. The torsion part and the free part. Let $\mathbb{A}$ be a PID and let M be an $\mathbb{A}$-module given by a finite number of generators and relations.

(a) Carefully state the Krull-Schmidt theorem for M.

(b) Prove that if K is a submodule of M then $\mathrm{Tor}(K) \subseteq \mathrm{Tor}(M)$.

(c) Prove that M is free of finite rank if and only if $\mathrm{Tor}(M) = 0$.

(d) Prove that if M is free of finite rank and K is a submodule of M then K is free of finite rank.