1.4 Tutorial 2 MAST30005, Semester I, 2024: Fields and vector spaces

1. Carefully define a field.
2. Carefully define a vector space.
3. Carefully define span(S).
4. Carefully define linearly independent.
5. Carefully define basis.
6. Carefully define $\mathbb{Q}$ and prove that it is a field.
7. Carefully define $\mathbb{C}$ and prove that it is a field.
8. Let $m \in \mathbb{Z}_{>0}$. Carefully define $\mathbb{Z}/m\mathbb{Z}$.
9. Let $p \in \mathbb{Z}_{>0}$. Show that $\mathbb{Z}/p\mathbb{Z}$ is a field if and only if p is prime.
10. Show that $3 \cdot 6 = 1 \cdot 6$ in $\mathbb{Z}/12\mathbb{Z}$.
11. Let $m \in \mathbb{Z}_{>1}$. Show that if m is not prime then there exist $a, b, c \in \mathbb{Z}/m\mathbb{Z}$ such that $ac = bc$ and $c \neq 0$ and $a \neq b$.
12. Let $\mathbb{F}$ be a field. Show that if $a, b, c \in \mathbb{F}$ and $ac = bc$ and $c \neq 0$ then $a = b$.
13. Show that if $a, b, c \in \mathbb{Z}$ and $ac = bc$ and $c \neq 0$ then $a = b$.
14. Carefully define $\mathbb{R}[x]$ and determine which of the axioms of a field it satisfies and which axioms of a field it does not satisfy.
15. Show that if $a, b, c \in \mathbb{R}[x]$ and $ac = bc$ and $c \neq 0$ then $a = b$.
16. Show that the $\mathbb{R}$-subspace of $\mathbb{C}$ with $\mathbb{R}$-basis $\{1, i\}$ is a field.
17. Show that the $\mathbb{Q}$-subspace of $\mathbb{C}$ with $\mathbb{Q}$-basis $\{1, i\}$ is a field.
18. Let $2^{1/3} \in \mathbb{R}_{\geq 0}$. Show that the $\mathbb{Q}$-subspace of $\mathbb{C}$ with $\mathbb{Q}$-basis $\{1, 2^{1/3}, 2^{2/3}\}$ is a field.
19. Let $\zeta = e^{2\pi i/3}$. Show that $\zeta^2 = -1 - \zeta$ and that the $\mathbb{Q}$-subspace of $\mathbb{C}$ with $\mathbb{Q}$-basis $\{1, \zeta, \}$ is a field.
20. Let $\zeta = e^{2\pi i/3}$. Show that $\zeta^2 = -1 - \zeta$ and that the $\mathbb{R}$-subspace of $\mathbb{C}$ with $\mathbb{R}$-basis $\{1, \zeta\}$ is a field.
21. Let $2^{1/3} \in \mathbb{R}_{\geq 0}$ and $\zeta = e^{2\pi i/3}$. Show that the $\mathbb{Q}$-subspace of $\mathbb{C}$ with $\mathbb{Q}$-basis $\{1, \zeta, 2^{1/3}, 2^{1/3}\zeta, 2^{2/3}, 2^{2/3}\zeta\}$ is a field.
22. Let $2^{1/3} \in \mathbb{R}_{\geq 0}$ and $\zeta = e^{2\pi i/3}$. Find a $\mathbb{Q}$-basis of the smallest field contained in $\mathbb{C}$ that contains $\mathbb{Q}$ and $2^{\frac{1}{3}}\zeta$.