1.13 Tutorial 11 MAST30005 Semester I, 2024

1. If R and S are rings, their product $R \times S = \{(r,s) \mid r \in R, s \in S\}$ is a ring with
$$(r,s) + (r',s') = (r+r', s+s') \qquad \text{and} \qquad (r,s)(r',s') = (rr', ss').$$
   (a) Write down the additive and multiplicative identities in $R \times S$.
   (b) Is $\mathbb{Z}/8\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (as rings)?
   (c) Is $\mathbb{Z}/6\mathbb{Z}$ isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (as rings)?

2. Let R and S be rings. Is the map
$$\text{Is} \quad \begin{matrix} R & \to & R \times S \\ r & \mapsto & (r,0) \end{matrix} \quad \text{a ring homomorphism?}$$
$$\text{Is} \quad \begin{matrix} R & \to & R \times R \\ r & \mapsto & (r,r) \end{matrix} \quad \text{a ring homomorphism?}$$

3. Let R be a ring. If I, J are ideals of R, the *sum* of I and J is defined by
$$I + J = \{x + y \mid x \in I, y \in J\} \subset R.$$
   (a) Show that $I + J$ is an ideal of R.
   (b) Prove the *Chinese Remainder Theorem*:
   If $I + J = R$ then $R/(I \cap J) \cong R/I \times R/J$ (as rings).
   (c) The classical Chinese remainder theorem says that if m and n are coprime integers, then for any a, b, the system of equations $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$ has a unique solution modulo mn. Show how this follows from the result called the Chinese remainder theorem above.

4. Find a greatest common divisor in $\mathbb{Z}[i]$ of $-1 + 7i$ and $18 - i$.

5. Let F be a field and $f(x) \in F[x]$ a polynomial such that $f(a) \neq 0$ for all $a \in F$. Show that if f has degree at most 3, then $f(x)$ is irreducible.

6. Find all irreducible polynomials of degree at most 3 in $\mathbb{F}_2[x]$. Show that $1 + x + x^4$ is irreducible in $\mathbb{F}_2[x]$.

7. Define $\mathbb{C}[t]$ and $\mathbb{C}[[t]]$.
   (a) Show that $\mathbb{C}[t]$ and $\mathbb{C}[[t]]$ are integral domains.
   (b) Determine $\mathbb{C}[t]^\times$ and $\mathbb{C}[[t]]^\times$.
   (c) Show that $\frac{1}{1-t} \in \mathbb{C}[[t]]$ and $e^t \in \mathbb{C}[[t]]$ and $\sin(t) \in \mathbb{C}[[t]$ and $\tan(t) \in \mathbb{C}[[t]]$.
   (d) Show that $t^{-1} \notin \mathbb{C}[[t]]$ and $\cot(t) \notin \mathbb{C}[[t]]$.
   (e) Let $\mathbb{C}(t)$ be the field of fractions of $\mathbb{C}[t]$ and let $\mathbb{C}((t))$ be the field of fractions of $\mathbb{C}[[t]]$. Show that
$$\mathbb{C}((t)) = \{0\} \cup \Big(\bigsqcup_{i \in \mathbb{Z}} t^i \mathbb{C}[[t]]^\times \Big).$$

8. Let R be a nonzero ring. An element $a \in R$ is nilpotent if there exists $n \in \mathbb{Z}_{>0}$ such that $a^n = 0$. Let $a \in R$. Prove that if a is nilpotent then $1 + a$ is a unit.

9. Let a and b be integers with $\gcd(a,b) = 1$ (in $\mathbb{Z}$). Prove that the greatest common divisor of a and b in $\mathbb{Z}[i]$ is also 1.