16.4 Problem Sheet: Modules

1. (a) Let R be a principal ideal domain and I a nonzero ideal of R. Prove that there are only finitely many ideals J with $J \supset I$.

   (b) Give an example of a unique factorisation domain R and a nonzero ideal I of R for which there are infinitely many ideals J with $J \supset I$.

2. Prove that the polynomial $x^5 - 7x^4 - 3$ is irreducible in $\mathbb{Q}[x]$.

3. Let N be the submodule of the $\mathbb{Z}$-module $\mathbb{Z}^3$ generated by the vectors

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 9 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 12 \\ 3 \end{pmatrix}.$$

   Find a basis $\{b_1, b_2, b_3\}$ of $\mathbb{Z}^3$, and $d_1, d_2, d_3 \in \mathbb{Z}$, such that the nonzero elements in the set $\{d_1b_1, d_2b_2, d_3b_3\}$ form a basis for N.

4. Let A be a $m \times n$ matrix. Then A determines a $\mathbb{Z}$-module homomorphism

$$f_A : \mathbb{Z}^n \to \mathbb{Z}^m$$

   by $f_A(\mathbf{v}) = A\mathbf{v}$, with elements of $\mathbb{Z}^n$ and $\mathbb{Z}^m$ written as column vectors.

   Similarly, the transpose A^T of A determines a $\mathbb{Z}$-module homomorphism

$$f_{A^T} : \mathbb{Z}^m \to \mathbb{Z}^n.$$

   Prove the cokernels of f_A and f_{A^T} have isomorphic torsion subgroups. [Recall the cokernel of $\phi : M \to N$ is defined as the quotient $N/\mathrm{im}(\phi)$.]

5. Let k be an algebraically closed field. Let V be a finite dimensional k-vector space and let $T : V \to V$ be a linear transformation. Define

$$A = \bigcup_{i=1}^{\infty} \ker(T^i), \qquad \text{and} \qquad B = \bigcap_{i=1}^{\infty} \mathrm{im}(T^i).$$

   Prove that A and B are subspaces of V, and that $V \cong A \oplus B$.

6. Let M be an R-module. Show that for all $r \in R$ and $m \in M$ we have

   (a) $0m = 0$

   (b) $r0 = 0$

   (c) $(-r)m = -(rm) = r(-m)$.

7. Let R be a ring and $a_1, \ldots, a_n \in R$. Let $M = \{(x_1, x_2, \ldots, x_n) \in R^n \mid a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0\}$. Prove that M is a submodule of R^n.

8. Let $N = \{(x, y, z) \in \mathbb{Z}^3 \mid x + y + z = 0\}$. Is N a free $\mathbb{Z}$-module? If so, find a basis.

9. Let $\phi : M \to N$ be a homomorphism of R-modules that is a bijection. Prove that $\phi^{-1} : N \to M$ is also a homomorphism.

10. Show that $\mathbb{Q}$ is not free as a $\mathbb{Z}$-module (remember the basis may be infinite).

11. (a) Let V be a vector space over a field k. Let $T : V \to V$ be a linear transformation. Show that by defining $(\sum_i a_i x^i) \cdot v = \sum_i a_i T^i(v)$ defines the structure of a $k[x]$-module on V.
    (b) Find an example of a vector space V, together with two linear transformations T and S, such that there does not exist a $k[x,y]$-module structure on V with $x \cdot v = T(v)$ and $y \cdot v = S(v)$ for all $v \in V$.

12. Let M be an R-module and N be a submodule of M. Find a natural bijection between submodules of M/N and submodules of M containing N. (This result sometimes goes by the name of the correspondence theorem)

13. Let R be a principal ideal domain, $p \in R$ an irreducible element, $k \geq 1$ and let M be the R-module $R/(p^k)$. Let $N = p^{k-1}M := \{p^{k-1}m \mid m \in M\}$.
    (a) Show that N is a submodule of M.
    (b) Show that N is contained in every non-zero submodule of M.
    (Hint(??): Consider the surjective homomorphism $R \to M$, $a \mapsto a + (p^k)$.)

14. An R-module is called *cyclic* if it has a generating set with one element.
    (a) Is a quotient of a cyclic module necessarily cyclic?
    (b) Is a submodule of a cyclic module necessarily cyclic?

15. Let $f : N \to M$ be an injective homomorphism of R-modules. Suppose that there exists a homomorphism $\pi : M \to N$ such that $\pi(f(n)) = n$ for all $n \in N$. Prove that $M \cong N \oplus X$ for some R-module X.

16. A module M is called *Noetherian* if for every sequence of submodules of M

$$N_0 \subset N_1 \subset N_2 \subset N_3 \subset \cdots$$

there exists k with $N_k = N_{k+1} = N_{k+2} = \cdots$.
    (a) Show that a submodule of a Noetherian module is Noetherian.
    (b) Show that a quotient of a Noetherian module is Noetherian.
    (c) Show that if M' is a submodule of M, and if M' and M/M' are both Noetherian, then M is Noetherian.
    (d) Show that a Noetherian module is finitely generated.
    (e) Can you use this to prove that the torsion submodule of a finitely generated module over a principal ideal domain is Noetherian?

17. Let $R = \mathbb{Z}/(p^2)$. Let M be a finite R-module. Suppose there is an injective R-module homomorphism $\iota : R \to M$. Prove that there exists a R-module homomorphism $\pi : M \to R$ such that $\pi \circ \iota(r) = r$ for all $r \in R$. How far can you generalise this?

18. Let

$$A = \begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

    (a) Find the Smith Normal form (over $\mathbb{Z}$) of the matrix A.

(b) If M is a $\mathbb{Z}$-module with presentation matrix A, then show that $M \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.

(c) If N is a $\mathbb{Q}$-module with presentation matrix A, identify N.

19. Let V be a two dimensional vector space over $\mathbb{Q}$ having basis $\{v_1, v_2\}$. Let T be the linear operator on V defined by $T(v_1) = 3v_1 - v_2$, $T(v_2) = 2v_2$. Recall V (together with T) can be identified with a $\mathbb{Q}[t]$-module by defining $tu = T(u)$.

    (a) Show that the subspace $U = \{av_2 \mid a \in \mathbb{Q}\}$ of V spanned by v_2 is actually a $\mathbb{Q}[t]$-submodule of V.

    (b) Consider the polynomial $f = t^2+2t-3$. Determine the vectors fv_1 and fv_2, that is, express them as linear combinations of v_1 and v_2.

20. Let X be a $n \times m$ matrix with entries in a ring R. Define an ideal $d_1(X)$ to be the ideal in R generated by all entries of X. Let A and B be invertible matrices (of the appropriate sizes) with entries in R. Prove that $d_1(AXB) = d_1(X)$.

21. Let M be an R-module. Suppose that U and V are two submodules of M. Show that $M \cong U \oplus V$ if and only if $U \cap V = \{0\}$, and $U + V = M$. [The definition of $U + V$ is $U + V = \{u + v \mid u \in U, v \in V\}$]

22. Show that the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime and n a positive integer, is not a direct sum of two non-zero $\mathbb{Z}$-modules.

23. Up to isomorphism, how many abelian groups of order 96 are there?

24. With notation as in Question 20, let $d_k(X)$ be the ideal in R generated by all $k \times k$ minors in X. Prove that $d_k(AXB) = d_k(X)$.

25. Use the previous result to show that the elements d_i in Smith Normal Form are unique up to associates.

26. Find an isomorphic direct sum of cyclic groups, where V is an abelian group generated by x, y, z and subject to relations:

    (a) $3x + 2y + 8z = 0$, $2x + 4z = 0$

    (b) $x + y = 0$, $2x = 0$, $4x + 2z = 0$, $4x + 2y + 2z = 0$

    (c) $2x + y = 0$, $x - y + 3z = 0$

    (d) $4x + y + 2z = 0, 5x + 2y + z = 0, 6y - 6z = 0$.

27. Let V be the $\mathbb{Z}[i]$-module $(\mathbb{Z}[i])^2/N$ where

$$N = \mathrm{span}_{\mathbb{Z}[i]}\{(1 + i, 2 - i), (3, 5i)\}.$$

    Write V as a direct sum of cyclic modules.

28. Let $\mathbb{F}$ be a field and define $D\colon \mathbb{F}[x] \to \mathbb{F}[x]$ by

$$D(a_0 + a_1x + \cdots + a_nx^n) = a_1 + 2a_2x + \cdots + na_nx^{n-1},$$

    where $m = \underbrace{1 + \cdots + 1}_{m} \in \mathbb{F}$.

    (a) Verify that $D(fg) = D(f)g + fD(g)$, for all $f, g \in \mathbb{F}[x]$.

(b) An element α is called a double root of f if $(x-\alpha)^2$ divides f. Prove that α is a double root of f if and only if $f(\alpha)=0$ and $(Df)(\alpha)=0$.

29. Let $E=\mathbb{Q}(\alpha)$, where $\alpha^3-\alpha^2+\alpha+2=0$. Express $(\alpha^2+\alpha+1)(\alpha^2-\alpha)$ and $(\alpha-1)^{-1}$ in the form $a\alpha^2+b\alpha+c$ with $a,b,c\in\mathbb{Q}$.

30. Given the matrix $A=\begin{pmatrix}1-x & 1+x & x\\ x & 1-x & 1\\ 1+x & 2x & 1\end{pmatrix}\in M_{3\times 3}(R)$, $R=\mathbb{Q}[x]$, determine the R-module V presented by A. Is V a cyclic R-module? (A module is said to be *cyclic* if it is generated by a single element).

31. (a) Compute the characteristic polynomial of the following matrix: [as a reminder, the characteristic polynomial of a matrix A is $\det(\lambda I-A)$, which is a polynomial in the variable λ]

$$\begin{pmatrix}0 & 0 & 0 & \cdots & 0 & -a_0\\ 1 & 0 & 0 & \cdots & 0 & -a_1\\ 0 & 1 & 0 & \cdots & 0 & -a_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1}\end{pmatrix}$$

(b) What is the characteristic polynomial of any matrix in rational canonical form?

(c) Use this to prove the Cayley-Hamilton Theorem: If A is a square matrix and $p(t)$ is its characteristic polynomial, then $p(A)=0$. [The Cayley-Hamilton theorem holds for matrices with entries in an arbitrary ring, but the intent of this question is to prove it for matrices with entries in a field. However, we can reduce the ring case to the field case (remember how we said to prove $\det(AB)=\det(A)\det(B)$, we could say WLOG R was a field of characteristic zero)]

32. Let $R=\mathbb{Q}[x]$ and suppose that the R-module M is a direct sum of four cyclic modules

$$\mathbb{Q}[x]/((x-1)^3)\oplus\mathbb{Q}[x]/((x^2+1)^2)\oplus\mathbb{Q}[x]/((x-1)(x^2+1)^4)\oplus\mathbb{Q}[x]/((x+2)(x^2+1)^2).$$

(a) Decompose M into a direct sum of cyclic modules of the form $\mathbb{Q}[x]/(f_i^{m_i})$, where f_i are monic irreducible polynomials in $\mathbb{Q}[x]$ and $m_i>0$.

(b) Find $d_1,d_2,\ldots,d_k\in\mathbb{Q}[x]$ monic polynomials with positive degree such that $d_i|d_{i+1}$, $i=1,\ldots,k-1$ and $M\cong\mathbb{Q}[x]/(d_1)\oplus\cdots\oplus\mathbb{Q}[x]/(d_k)$.

(c) Identify the $\mathbb{Q}[x]$-module M with the vector space M over $\mathbb{Q}$ together with a linear operator $X:M\to M, v\mapsto xv$. Suppose the matrix of X is A with respect to a $\mathbb{Q}$-vector space basis of M. Determine the minimal and characteristic polynomials of A and the dimension of M over $\mathbb{Q}$. (the *minimal polynomial* of A is the smallest degree monic polynomial $f(x)\in\mathbb{Q}[x]$ such that $f(A)=0$.)

33. Let $V=\mathbb{C}[t]/((t-\lambda)^m)$, $\lambda\in\mathbb{C}$, $m>0$, be a cyclic $\mathbb{C}[t]$-module.

(a) Show that

$$(w_0=\bar{1}, w_1=\overline{t-\lambda}, w_2=\overline{(t-\lambda)^2},\ldots,w_{m-1}=\overline{(t-\lambda)^{m-1}})$$

is a basis of V as $\mathbb{C}$-vector space.

(b) Show that the matrix of $T: V \to V, v \mapsto tv$ with respect to the basis in (a) is of the form

$$A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{pmatrix} \in M_{m\times m}(\mathbb{C}).$$

34. Suppose that V is an 8 dimensional complex vector space and $T: V \to V$ is a linear operator. Using T we make V into a $\mathbb{C}[t]$-module in the usual way. Suppose that as a $\mathbb{C}[t]$-module

$$V \cong \mathbb{C}[t]/((t+5)^2) \oplus \mathbb{C}[t]/((t-3)^3(t+5)^3).$$

What is the Jordan (normal) form for the transformation T? What are the minimal and characteristic polynomials of T?

35. Let V be an $F[t]$-module and $(v_1, \ldots, v_n)$ a basis of V as an F-vector space. Let $T: V \to V$ be a linear operator and $A \in M_{n\times n}(F)$ the matrix of T with respect to the basis $(v_1, \ldots, v_n)$. Prove that the $F[t]$-matrix $tI - A$ is a presentation matrix of (V, T) regarded as a $F[t]$-module.

36. Determine the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{C})$ by decomposing the $\mathbb{C}[t]$-module V presented by the matrix $tI - A \in M_{3\times 3}(\mathbb{C}[t])$.

37. Find all possible Jordan normal forms for a matrix $A \in M_{5\times 5}(\mathbb{C})$ whose characteristic polynomial is $(t+2)^2(t-5)^3$.

38. Let M be an R-module. Show that $N \subseteq M$ is a submodule if and only if

    (a) N is nonempty,
    (b) If $n_1, n_2 \in N$ then $n_1 + n_2 \in N$,
    (c) If $n \in N$ and $c \in R$ then $cn \in N$.

39. Let $\varphi\colon V \to W$ be an R-module homomorphism. Shoe that $\ker(\varphi)$ is a submoduel of V and that $\mathrm{im}(\varphi)$ is a submodule of W.

40. Let M be an R-module.

    (a) If $m \in M$ then $0m = 0$,
    (b) if $r \in R$ then $r0 = 0$,
    (c) if $r \in R$ and $m \in M$ then $(-r)m = -(rm) = r(-m$.

41. State and prove module versions of the three isomorphism theorems, and the correspondence theorem.

42. Let R be a ring and let V be a free module of finite rank over R.

    (a) Show that every set of generators of V contains a basis of V.
    (b) Show that every linearly independent set in V can be extended to a basis of V.

43. Llet M be an R-module. Suppose that U and V are two submodules of M satisfying $U \cap V = \{0\}$ and $U + V = M$. Show that $M \cong U \oplus V$.

44. Let U and V be R-modules and $M = U \oplus V$. Define submodules U' and V' of M by

$$U' = \{(u, 0) \mid u \in U\} \qquad \text{and} \qquad V' = \{(0, v) \mid v \in V\}.$$

Show that $U' \cap V' = \{0\}$, $U' + V' = M$ and $U' \cong U$ and $V' \cong V$.

45. Show that if M_1, M_2, N_1, N_2 are R-modules then

$$\frac{M_1 \oplus M_2}{N_1 \oplus N_2} \cong \frac{M_1}{N_1} \oplus \frac{M_2}{N_2}.$$

46. Let R be a PID. Let $p \in R$ be irreducible, $k \in \mathbb{Z}_{\geq 1}$ and $M = \frac{R}{p^k R}$. Let $N = p^{k-1}M$.
    (a) Show that N is a submodule of M.
    (b) Show that N is contained in every non-zero submodule of M.

47. Show that $R\text{-span}(S) = \{r_1 v_1 + \cdots r_k v_k \mid k \in \mathbb{Z}_{>0},\ r_1, \ldots, r_k \in R \text{ and } v_1, \ldots, v_k \in S\ \}$.

48. Let M be an R-module. Prove that a subset S of M is a basis of M if and only if every element of M can be written uniquely as a linear combination of elements from S.

49. Let R be an integral domain. Let I be an ideal in R. Show that I is a free R-module if and only if it is principal.

50. Let F and G be two free R-modules of rank m and n respectively. Shwo that the R-module $F \oplus G$ is free of rank $m + n$.

51. Show that if N and M/N are finitely generated as R-modules then M is also a finitely generated R-module.

52. Prove that $\mathbb{Q}$ is not finitely generated as a $\mathbb{Z}$-module.

53. Show that a quotient of a cyclic module is cyclic.

54. Show that a submodule of a cyclic module is cyclic.

55. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(0, 3)\}$.
    Write M/N as a direct sum of cyclic submodules.

56. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(2, 0), (0, 3)\}$.
    Write M/N as a direct sum of cyclic submodules.

57. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(2, , 3)\}$.
    Write M/N as a direct sum of cyclic submodules.

58. Let $M = \mathbb{Z} \oplus \mathbb{Z}$ and let $N = \mathbb{Z}\text{-span}\{(6, 9)\}$.
    Write M/N as a direct sum of cyclic submodules.

59. Let V be a two dimensional vector space over $\mathbb{Q}$ having basis $\{v_1, v_2\}$. Let T be the linear transformation on V defined by $T(v_1) = 3v_1 - v_2$ and $T(v_2) = 2v_2$. Make V into a $\mathbb{Q}[X]$-module by defining $Xu = T(u)$.
    (a) Show that the subspace $U = \{av_2 \mid a \in \mathbb{Q}\}$ is a $\mathbb{Q}[X]$-submodule of V.
    (b) Let $f = X^2 + 2X - 3 \in \mathbb{Q}[X]$. Determine the vectors fv_1 and fv_2 as linear combinations of v_1 and v_2.

60. Let M be an R-module and let $m \in M$. Show that $\text{ann}(m)$ is an ideal in R.

61. Let M be an R-module. Show that $\text{Tor}(M)$ is a submodule of M.

62. Let R be a integral domain and let M be a free R-module. Show that M is torsion free.

63. Give an example of an integral domain R and an R-module M such that M is torsion free and M is not free.

64. Let I be an ideal in R. Show that $\mathrm{ann}(R/I) = I$.

65. Let M_1 and M_2 be R-modules. Show that $\mathrm{ann}(M_1 \oplus M_2) = \mathrm{ann}(M_1) \oplus \mathrm{ann}(M_2)$.

66. Show that R is a torsion free R-module if and only if R is an integral domain.

67. Show that $\mathbb{Q}$ as a $\mathbb{Z}$-module is torsion free but not free.

68. Let R be a PID. Let M be a simple R-module. Show that either R is a field and $M \cong R$ or R is not a field and $M \cong R/pR$ for some prime $p \in R$.

69. Let $R = \mathbb{Z}/6\mathbb{Z}$ and let $F = R^{\oplus 2}$. Write down a basis of F. Let $N = \{(0,0), (3,0)\}$. Show that N is a submodule of the free module F and N is not free.

70. Let $R = \mathbb{Z}$ and $F = \mathbb{Z}^3$. Let $N = \{(x,y,z) \in F \mid x+y+z = 0\}$. Show that N is a submodule of F and find a basis of N.

71. Let $A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$. Find $L, R \in GL_3(\mathbb{Z})$ and $d_1, d_2, d_3 \in \mathbb{Z}_{\geq 0}$ such that $d_3\mathbb{Z} \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2, d_3)$.

72. Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$. Find $L, R \in GL_2(\mathbb{Z})$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2)$.

73. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Find $L \in GL_2(\mathbb{Z})$ and $R \in GL_3(\mathbb{Z})$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2)$.

74. Let $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$. Find $L, R \in GL_3(\mathbb{Z})$ and $d_1, d_2, d_3 \in \mathbb{Z}_{\geq 0}$ such that $d_3\mathbb{Z} \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2, d_3)$.

75. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} 1-X & 1+X & X \\ X & 1-X & 1 \\ 1+X & 2X & 1 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

76. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} X & 1 & -2 \\ -3 & X+4 & -6 \\ -2 & 2 & X-3 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

77. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} X & 0 & 0 \\ 0 & 1-X & 0 \\ 0 & 0 & 1-X^2 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

78. . Let R be an integral domain. Let V be a free R-module of rank d. Define $\mathrm{End}_R(V)$, explain (with proof) how it is a ring, and show that $\mathrm{End}_R(V) \cong M_{d\times d}(R)$.

79. Let R be an integral domain. Let V be a free R-module with basis $\{v_1, \ldots, v_d\}$. Let $\varphi\colon V \to V$ be an R-module morphism. Prove that $\{\varphi(v_1), \ldots, \varphi(v_d)\}$ is a basis of V if and only if φ is an isomorphism.

80. Let $A \in M_{d\times d}(\mathbb{Z})$ and let φ be the $\mathbb{Z}$-module morphism given by

$$\begin{array}{cccc} \varphi\colon & \mathbb{Z}^k & \to & \mathbb{Z}^k \\ & v & \mapsto & Av. \end{array} \qquad \text{Show that} \quad \mathrm{Card}\left(\frac{\mathbb{Z}^k}{\mathrm{im}(\varphi)}\right) = \begin{cases} |\det(A)|, & \text{if } \det(A) \neq 0, \\ \infty, & \text{if } \det(A) = 0. \end{cases}$$

81. Let V be the $\mathbb{Z}[i]$-module $(\mathbb{Z}[i])^2/N$, where $N = \mathbb{Z}[i]\text{-span}\{(1+i, 2-i), (3, 5i)\}$. Write V as a direct sum of cyclic modules.

82. Let $p \in \mathbb{Z}_{>0}$ prime and let $n \in \mathbb{Z}_{>\geq 0}$. Show that the $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$ is not a direct sum of two nontrivial $\mathbb{Z}$-modules.

83. Let $R = \mathbb{Q}[X]$ and suppose that the rotsion R-module M is a direct sum of four cyclic modules whose annihilators are $(X-1)^3$, $(X^2+1)^3$, $(X-1)(X^2+1)^4$ and $(X+2)(X^2+1)^2$. Determine the primary decomposition of M and the invariant factor decomposition of M. If M is thought of as a $\mathbb{Q}$-vector space on which X acts as a linear transformation denoted A, determine the mninimal and the characteristic polynomials of A and the dimension of M over $\mathbb{Q}$.

84. How many abelian groups of order 136 are there? Give the primary and invariant factor decompositions of each.

85. Determine the invariant factors of the abelaingroup $C_{100} \oplus C_{36} \oplus C_{150}$.

86. Find a direct sum of cyclic groups which is isomorphic to the abeliangroup $\mathbb{Z}^3/N$, where N is generated by $\{(2, 2, 2), (2, 2, 0), (2, 0, 2)\}$.

87. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad 3x + 2y + 8z = 0 \text{ and } 2x + 4z = 0.$$

88. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad x + y = 0,\ 2x = 0,\ 4x + 2z = 0 \text{ and } 4x + 2y + 2z = 0.$$

89. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad 2x + y = 0 \text{ and } x - y + 3z = 0.$$

90. Find an isomorphic direct product of cyclic groups and the invariant factors of V, where V is an abeliangroup

$$\text{generated by } x, y, z \quad \text{with relations} \quad 4x + y + 2z = 0,\ 5x + 2y + z = 0 \text{ and } 6 - 6z = 0.$$

91. Let V be a $\mathbb{C}$-vector space with $\dim(V) = 8$ and $T: V \to V$ a linear transformation. Suppose that, as a $\mathbb{C}[t]$-module
$$V \cong \frac{\mathbb{C}[t]}{(t+5)^2\mathbb{C}[t]} \oplus \frac{\mathbb{C}[t]}{(t-3)^3(t+5)^3\mathbb{C}[t]}.$$
What is the Jordan normal form for the transformation T? What are the eigenvalues of T and how many eigenvectors does T have? What are the minimal and characteristic polynomials of T?

92. Determine the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ by calculating the invariant factor matrix of $X - A$.

93. Find all possible Jordan normal forms for a matrices with characteristic polynomial $(t+2)^2(t-5)^3$.

94. Find the Smith normal form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Z}$.

95. Find the rational canonical form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Q}$.

96. Find the Jordan canonical form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{C}$.

97. Find the Smith normal form of $\begin{pmatrix} 11 & -4 & 7 \\ -1 & 2 & 1 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Z}$.

98. Let R be a PID and let $a, b \in R$. Prove that $a = bc$ implies $aR \subseteq bR$.

99. Let R be a PID and let $a, b \in R$. Prove that $aR = R$ implies that a is a unit.

100. Let R be a PID and let $a, b \in R$. Prove that $aR = bR$ implies that there exists $u \in R^\times$ such that $au = b$.

101. Let R be a PID and let $a, b \in R$. What is the definition of $gcd(a, b)$? Using your definition state and prove Bezout's Lemma in a PID.

102. Let R be a PID, let K be a field and let $\phi: R \to K$ be a ring homomorphism. Prove that the kernel of ϕ is a prime ideal.

103. Let R be a PID, let K be a field and let $\phi: R \to K$ be a ring homomorphism. Prove that the image of ϕ is either a field or is isomorphic to R.

104. Let A be the abelian group with generators a, b, c and relations
$$3a = b - c, \qquad 6a = 2c, \qquad 3b = 4c.$$
   (a) Find the generator mmatrix for A.
   (b) Bring this matrix into Smith normal form, carefully recording each step.

(c) By the classification theorem for finitely generated abelain groups, A is isomorphic to a cartesian product of cyclic groups of prime power order and/or copies of $\mathbb{Z}$. Which group is it?

(d) Write down an explicit isomorphism from A to the group you have identified in part (c).

(e) Interpret the meaning of the three matrices L, D and R in this context.

105. Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{pmatrix}$.

(a) Use whichever method you prefer to bring A into Jordan normal form. Carefully record your steps.

(b) Recall how to use A to equip $\mathbb{C}^3$ with the sturcture of a $\mathbb{C}[x]$-module.

(c) Write down generators and relations for the $\mathbb{C}[x]$ module encoded by A.

(d) The structure theorem for module over a PID gives you a different (potentially smaller) set of generators and relations. What is it in this example?

(e) Find an explicit isomorphism between the representations of parts (c) and (d).

106. Let R be a PID and let F and G be free R-modules of finite rank. Let $\varphi\colon F \to G$ be and R-module morphism. Show that the rank of $\mathrm{im}(\varphi)$ is bounded bove by the rank of F.

107. Let M be a R-module such that all R-submodules are finitely generated. Show that M satisfies ACC.

108. Let M be the $\mathbb{Q}[x]$-module given by

$$M = \frac{\mathbb{Q}[x]}{(x^2+x+1)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x^3-1)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x-3)^2\mathbb{Q}[x]}.$$

Let $T\colon M \to M$ be the $\mathbb{Q}$-linear transformation given by $T(u) = Xu$.

(a) Give the primary decomposition of M as a $\mathbb{Q}[x]$-module.

(b) What is the dimension of M as a vector space over $\mathbb{Q}$?

(c) What is the minimal polynomial of T?

109. Let M be the $\mathbb{C}[x]$-module given by

$$M = \frac{\mathbb{C}[x]}{(x^2+x+1)\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x^3-1)\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x-3)^2\mathbb{C}[x]}.$$

Let $T\colon M \to M$ be the $\mathbb{C}$-linear transformation given by $T(u) = Xu$.

(a) Give the primary decomposition of M as a $\mathbb{C}[x]$-module.

(b) What is the Jordan normal form matrix for T?

110. . Let $\varphi\colon \mathbb{Z}^4 \to \mathbb{Z}^3$ be the $\mathbb{Z}$-module homomorphism determined by

$$\varphi(1,0,0,0) = (14,8,2), \quad \varphi(0,1,0,0) = (12,6,0), \quad \varphi(0,0,1,0) = (18,12,0), \quad \varphi(0,0,0,1) = (0,6,0).$$

(a) Find bases for $\mathbb{Z}^3$ and $\mathbb{Z}^4$ such that the matrix of φ with respect to these bases is in Smith normal form.

(b) Find the invariant factor decomposition of the $\mathbb{Z}$-module $\mathbb{Z}^4/\mathrm{im}(\varphi)$.

111. Define carefully what it means for an R-module to be finitely generated.

112. Let M and N be R-modules. Show that if N and M/N are finitely generated then M is finitely generated.

113. Give an example of a $\mathbb{Z}$-module that is not finitely generated.

114. Let M be the $\mathbb{Z}[i]$-module given by $M = \mathbb{Z}[i]^3/N$ where N is the submodule of $\mathbb{Z}[i]^3$ generated by

$$\{(-i,0,0),(1-2i,0,1+i),(1+2i,-2i,1+3i)\}.$$

If

$$A = \begin{pmatrix} -i & 1-2i & 1+2i \\ 0 & 0 & -2i \\ 0 & 1+i & 1+3i \end{pmatrix}, \qquad X = \begin{pmatrix} i & 2 & 0 \\ 0 & 0 & -i \\ 0 & i & 1+i \end{pmatrix}, \qquad Y = \begin{pmatrix} -i & i & 1+i \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$X^{-1}AY = \begin{pmatrix} i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Write M as a direct sum of nontrivial cyclic $\mathbb{Z}[i]$-modules.

(b) Calculate the annihilator of M.

(c) Find an element $u \in M - \{0\}$ with the property that $iu = u$.

115. Let V be a complex vector space of dimension 9 and let $T\colon V \to V$ be a linear transformation. Explain how T can be used to make V into a $\mathbb{C}[X]0$-module.

116. Let V be a complex vector space of dimension 9 and let $T\colon V \to V$ be a linear transformation. Suppose that, as a $\mathbb{C}[X]$-module,

$$V \cong \frac{\mathbb{C}[X]}{(X-5)^2(X+2)^2} \oplus \frac{\mathbb{C}[X]}{(X+5)^2(X+2)^2}.$$

(i) What is the Jordan normal form of T?

(ii) What are the minimal and characteristic polynomials of T?

117. State the structure theorem for finitely generated modules over a principal ideal domain.

118. Let V be the $\mathbb{Q}[X]$-module given by $V = \mathbb{Q}[X]^4/N$ where N is the submodule of $\mathbb{Q}[X]^4$ generated by

$$\{(1,0,1,0),(1,X,0,0),(1,0,-X,0),(-1,0,1,x^2)\}.$$

(i) Find the invariant factor decomposition of V.

(ii) Write down the primary decomposition of V.

119. (a) Give the definitions of a module and a free module.

(b) Give an example of a free module having a proper submodule of the same rank.

(c) Show that, as a $\mathbb{Z}$-module, $\mathbb{Q}$ is torsion free but not free.

120. State the structure theorem for finitely generated modules over a PID.

121. (a) Let $N \subseteq \mathbb{Z}^3$ be the submodule generated by the set $\{(2,4,1),(2,-1,1)\}$. Find a basis $\{f_1, f_2, f_3\}$ for $\mathbb{Z}^3$, and elements $d_1, d_2, d_3 \in \mathbb{Z}$ such that the nonzero elements $\{d_1 f_1, d_2 f_2, d_3 f_3\}$ form a basis for N and $d_1 | d_2 | d_3$.
   (b) Write $\mathbb{Z}^3/N$ as a direct sum of nontrivial cyclic $\mathbb{Z}$-modules.

122. Explain why there is no $u \in \mathbb{Z}^3$ such that $\{(2,4,1),(2,-1,1),u\}$ is a basis of $\mathbb{Z}^3$.

123. Let V be an 8-dimensional complex vector space and let $T\colon V \to V$ be a linear transformation. Explain how V can be regarded as a $\mathbb{C}[X]$-module.

124. Let V be the $\mathbb{C}[X]$-modules given by

$$V = \frac{\mathbb{C}[X]}{(X-2)(X-3)^2} \oplus \frac{\mathbb{C}[X]}{(X-2)(X-3)^3}.$$

Let $T\colon V \to V$ be the linear transformation determined by the action of T.

   (i) What is the Jordan Normal Form of T?
   (ii) What is the minimal polynomial of T?
   (ii) What is the dimension of the eigenspace of T correspondng to the eigenvalue 3?

125. Define what it means to say that a module is torsion free.

126. Let R be an integral domain. Show that a free R-module is torsion free..

127. Give an example of a finitely generated R-module that is torsion-free but not free.

128. Let R be a commutative unital ring, let F be a free R-module and let $\varphi\colon M \to F$ be a surjective module homomorphism. Show that $M \cong F \oplus \ker(\varphi)$.

129. State the structure theorem for finitely generated modules over a PID.

130. Let M be the $\mathbb{Z}$-module given by $M = \mathbb{Z}^4/N$, where N is the submodule of $\mathbb{Z}^4$ generated by $\{(15,1,8,1),(0,2,0,2),(7,1,4,1)\}$.
   (i) Write M as a direct sum of non-trivial cyclic $\mathbb{Z}$-modules.
   (ii) What is the torsion-free rank of M?

131. Let V be a finite dimensional real vector space and let $T\colon V \to V$ be a linear transformation. View V as an $\mathbb{R}[X]$-module. Show that V is finitely generated and is a torsion module.

132. Assume that

$$M \cong \frac{\mathbb{R}[X]}{(X^2+1)^2(X-2)} \oplus \frac{\mathbb{R}[X]}{(X^2-1)^2} \oplus \frac{\mathbb{R}[X]}{(X-1)}.$$

   (i) What is the primary decomposition of M?
   (ii) What is the dimension of V as a real vector space?
   (iii) What is the minimal polynomial of T?

133. Let M be a module. Define carefully what it means to say that M is free.

134. Give an example of a submodule of a free module that is not free.

135. Give an example of a free module M and a generating set $S \subseteq M$ such that M does not contain a basis.

136. Show that $\mathbb{Q}$, considered as a $\mathbb{Z}$-module, is not free.

137. State the structure theorem for finitely generated modules over a principal ideal domain.

138. Determine the invariant factor decomposition of the abelian group given by $\mathbb{Z}^3/N$ where N is the submodule of $\mathbb{Z}^3$ generated by $\{(7,4,1),(8,5,2),(7,4,1,5)\}$.

139. Let A be the abelian group given by $A = \mathbb{Z}^3/N$ where N is the submodule of $\mathbb{Z}^3$ generated by $\{(-4,2,6),(-6,2,6),(7,4,15)\}$. If

$$R = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 2 \\ 21 & 3 & 7 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 3 & -1 \\ 1 & -2 & 3 \\ 1 & 0 & 2 \end{pmatrix}$$

then

$$X^{-1}RY = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

(i) Find a basis $\{b_1, b_2, b_3\}$ of $\mathbb{Z}^3$ such that $\{d_1b_1, d_2b_2, d_3b_3\}$ generates N.

(ii) Find elements $u \in A$ and $v \in A$ that generate A and are such that $2u = 0$ and $6v = 0$.

140. Let $A \in M_{6\times 6}(\mathbb{C})$ such that $xI - A \in M_{6\times 6}(\mathbb{C}[x]$ is equivalent to the diagonal matrix $\mathrm{diag}(1,1,1,(x-2),(x-2),(x-2)^2(x-4)^2) \in M_{6\times 6}(\mathbb{C}[x])$.

(i) What is the Jordan normal form of A?

(ii) What are the characteristic and minimal polynomials of A?

141. Let V be the $\mathbb{R}[x]$ module given by

$$V = \frac{\mathbb{R}[x]}{(x-1)} \oplus \frac{\mathbb{R}[x]}{(x^2-2)} \oplus \frac{\mathbb{R}[x]}{(x^2+2)}.$$

(i) Calculate the primary decomposition of V.

(ii) Calculate the invariant factor decompositions of V.

(iii) What is the dimension of V when considered as a vector space over $\mathbb{R}$?

142. What does it mean to say that an R-module is free?

143. Let $R = \mathbb{R}[X,Y]$ and let $I = (X,Y)$ be the ideal generated by X and Y. Show that I considered as an R-module is not free.

144. Give the definition of the torsion submodule of an R-module.

145. Suppose that R is an integral domain and M is an R-module. Let T be the torsion submodule of M. Show that the R-module M/T is torsion free.

146. Let F be the $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule generated by

$$\{(4,-4,4),(-4,4,8),(16,20,4)\}.$$

Calculate the invariant factor decomposition of F/N.

147. Let F be the $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule generated by

$$\{(4,-4,4),(-4,4,8),(16,20,4)\}.$$

Calculate the primary decomposition of F/N.

148. Let F be the $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule generated by

$$\{(4,-4,4),(-4,4,8),(16,20,4)\}.$$

Find a basis $\{f_1,f_2,f_3\}$ for F and ingegers d_1,d_2,d_3 such that $d_1|d_2|d_3$ and $\{d_1f_1,d_2f_2,d_3f_3\}$ is a basis for N.

149. Let $A\in M_8(\mathbb{C})$. Explain how A can be used to define a $\mathbb{C}[X]$-module structure on $\mathbb{C}^8$.

150. Suppose that $XI-A\in M_8(\mathbb{C}[X])$ is equivalent to the matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & (X-1) & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & (X-1)(X-2)^2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (X-1)^2(X-2)^2
\end{pmatrix}.$$

(i) What is the Jordan normal form of A?

(ii) What are the minimal and characteristic polynomials of the matrix A?

151. Let M be an R-module. Give the definitions of what it means to say that M is torsion free and what it means to say that M is free.

152. Show that if R is an integral domain and M is free then M is torsion free.

153. Show that $\mathbb{Q}$ considered as a $\mathbb{Z}$-module, is torsion free but not free.

154. State the structure theorem for finitely generated modules over a PID.

155. Show that if R is a PID then any finitely generated and torsion free R module is free.

156. Let A be an abelian group with presentation

$$\langle a,b,c \mid 2a+b=0, 3a+3c=0\rangle.$$

Give the primary decomposition of A. What is the torsion free rank of A?

157. Calculate the invariant factor matrix over $\mathbb{Q}[x]$ for the matrix

$$\begin{pmatrix} 1 & x & -2\\ x+4 & -3 & -6\\ 2 & -2 & x-3\end{pmatrix}$$

158. Let V be an 8 dimensional complex vector space and $T\colon V\to V$ a linear transformation.

(i) Explain how T can be used to define a $\mathbb{C}[x]$-module structure on V.

(ii) Suppose that as a $\mathbb{C}[x]$ module

$$V \cong \frac{\mathbb{C}[x]}{(x-2)^2(x+3)^2} \oplus \frac{\mathbb{C}[x]}{(x-2)(x+3)^3}.$$

What is the Jordan normal form for the transformation T? What is the minimal polynomial of T?

159. State the structure theorem for finitely generated modules over a PID.

160. List, up to isomorphism, all abelian groups of order 360. Give the primary decomposition and the annihilator (as a $\mathbb{Z}$-module) of each group.

161. Let R be a commutative ring with identity. What does if mean to say that an R-module is free?

162. Show that every finitely generated R-module is isomorphic to a quotient of a free R-module.

163. Let F be the $\mathbb{Z}$-module $F = \mathbb{Z}^4$ and let N be the submodule of F generated by

$$\{(1,1,1,1), (1,-1,1,-1), (1,3,1,3)\}.$$

Give a direct sum of non-trivial cyclic $\mathbb{Z}$-modules that is isomorphic to F/N.

164. Let $A \in M_7(\mathbb{C})$ and suppose that the invariant factors of the matrix $xI - A \in M_7(\mathbb{C}[x])$ are $1, 1, 1, 1, x, x(x-i), x(x-i)^3$.

(a) Give the corresponding decomposition of $\mathbb{C}^7$ regarded as a $\mathbb{C}[x]$-module.

(b) Give the Jordan normal form of the matrix A.

(c) Give the minimal and characteristic polynomials of A.

(d) Is A diagonalizable?

165. List, up to isomorphism, all abelian groups of order 504. Give the primary decomposition and annihilator (as a $\mathbb{Z}$-module) of each group.

166. Let N be the submodules of the $\mathbb{Z}$-module $F = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by

$$\{(1,0,-1), (4,3,-1), (0,9,3), (3,12,3)\}.$$

(i) Find a basis $\{b_1, b_2, b_3\}$ of F and $d_1, d_2, d_3 \in \mathbb{Z}$ such that the non-zero elements of the set $\{d_1b_1, d_2b_2, d_3b_3\}$ form a basis for N (as a $\mathbb{Z}$-module).

(ii) Give a direct sum of non-trivial cyclic groups that is isomorphic to F/N.

167. Let $A \in M_8(\mathbb{C})$ be a matrix and suppose that the matrix $xI - A \in M_8(\mathbb{C}]x])$ is equivalent to the matrix

$$\mathrm{diag}(1, 1, 1, 1, (x-1), (x-1), (x-1)(x-2), (x-1)(x-2)^2(x-3)).$$

(a) Give the corresponding decomposition of $\mathbb{C}^8$ regarded as a $\mathbb{C}[x]$-module.

(b) Give the Jordan Normal form of the matrix A.

(c) Give the minimal and characteristic polynomials of A.

168. Exactly one of the following is a field:

$$\frac{\mathbb{Z}/3\mathbb{Z}[x]}{(x^2-2)} \qquad \text{and} \qquad \frac{\mathbb{Z}/7\mathbb{Z}[x]}{(x^2+3)}.$$

(i) Determine (with proof) which is a field and (with proof) which is not.

(ii) What is the order of the field F above?

(iii) Find a generator for the group $F^\times$ of nonzero elements of the field (under multiplication).

169. State the structure theorem for finitely generated modules over a PID.

170. Let $\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$ be the standard basis of the free $\mathbb{Z}$-module $\mathbb{Z}^3$ and let N be the submodule with basis

$$\mathcal{B} = \{(2,2,2),(0,8,4)\}.$$

Find a new basis $\{f_1, f_2, f_3\}$ for $\mathbb{Z}^3$ and elements $d_1, d_2, d_3 \in \mathbb{Z}$ such that the non-zero elements of the set $\{d_1f_1, d_2f_2, d_3f_3\}$ form a basis for N and $d_1|d_2|d_3$.

171. Find the invariant factor matrix over $\mathbb{Z}$ that is equivalent to the matrix

$$\begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$$

172. Give the primary decomposition and invariant factor decomposition of the $\mathbb{Z}$-module $\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/40\mathbb{Z} \oplus \mathbb{Z}/100\mathbb{Z}$.

173. Let V be an eight dimensional complex vector space and let $T\colon V \to V$ be a linear transformation. Explain how V can be regarded as a $\mathbb{C}[t]$-module.

174. Let V be an eight dimensional complex vector space and let $T\colon V \to V$ be a linear transformation. Suppose that

$$V \cong \frac{\mathbb{C}[t]}{(t-2)(1-3)^2} \oplus \frac{\mathbb{C}[t]}{(t-2)(t-3)^3}, \qquad \text{as a } \mathbb{C}[t]\text{-module.}$$

(i) What is the Jordan normal form of T?

(ii) What is the minimal polynomial of T?

(iii) What is the dimension of the eigenspace corresponding to the eigenvalue 3?

175. Determine which of the following abelian groups are isomorphic:

$$C_{12} \oplus C_{50} \oplus C_{30}, \quad C_4 \oplus C_{12} \oplus C_{15} \oplus C_{25}, \quad C_{20} \oplus C_{30} \oplus C_{30}, \quad C_6 \oplus C_{50} \oplus C_{60}.$$

176. Consider the abelian group A with generators x, y, z subject to the defining relations $7x + 5y + 2z = 0$, $3x + 3y = 0$ and $13x + 11y + 2z = 0$. Find a direct sum of cyclic groups which is isomorphic to A. Explain in sense your answer is unique.

177. Let M be a finitely generated torsion module over a PID R. Show that M is indecomposable if and only if $M = Rx$ where $\mathrm{ann}_R(z) = (p^e)$ and p is a prime of R.

178. Llet T be a linear operator on the finite dimensional vector space V over $\mathbb{C}$. Suppose that the characteristic polynomial of T is $(t+2)^2(t-5)^3$. Determine all possible Jordan forms for a matrix of T. In each case find the minimal polynomial for T and the dimension of the space of eigenvectors.

179. State carefully the invariant factor theorem which describes the structure of finitely generated modules over a principal ideal domain.

180. Describe the primary decomposition of a finitely generated torsion module over a PID.

181. Determine which if the following abelian groups are isomorphic:

$$C_6 \oplus C_{50} \oplus C_{60}, \quad C_{20} \oplus C_{30} \oplus C_{30}, \quad C_{12} \oplus C_{25} \oplus C_4 \oplus C_{15}, \quad C_{30} \oplus C_{50} \oplus C_{12}.$$

182. Let $R = \mathbb{Q}[x]$ and suppose that the torsion R-module M is a direct sum of four cyclic modules whose annihilators (order ideals) are $(x-1)^3$, $(x^2+1)^2$, $(x-1)(x^2+1)^4$ and $(x+2)(x^2+1)^2$. Determine the primary components and invariant factors of M.

183. Let $R = \mathbb{Q}[x]$ and suppose that the torsion R-module M is a direct sum of four cyclic modules whose annihilators (order ideals) are $(x-1)^3$, $(x^2+1)^2$, $(x-1)(x^2+1)^4$ and $(x+2)(x^2+1)^2$. If M is thought of as a vector space over $\mathbb{Q}$ on which x acts as a linear transformation denoted A, determine the minimum and characteristic polynomials of A and the dimension of M over $\mathbb{Q}$.

184. Let $R = \mathbb{C}[x]$ and suppose that the torsion R-module M is a direct sum of four cyclic modules whose annihilators (order ideals) are $(x-1)^3$, $(x^2+1)^2$, $(x-1)(x^2+1)^4$ and $(x+2)(x^2+1)^2$. If M is thought of as a vector space over $\mathbb{C}$ on which x acts as a linear transformation denoted A then is A diagonalizable?

185. Determine the invariant factors and the torsion free rank of the abelian group M generated by x, y, z subject to the relations

$$2x - 4y + 2z = 0 \qquad \text{and} \qquad -2x + 10y + 4z = 0.$$

186. Let M be the abelian group generated by x, y, z subject to the relations

$$2x - 4y + 2z = 0 \qquad \text{and} \qquad -2x + 10y + 4z = 0.$$

Express M as a direct sum of cyclic groups in a unique way.

187. Determine which of the following abelian groups are isomorphic:

$$C_6 \oplus C_{100} \oplus C_{15}, \qquad C_{50} \oplus C_6 \oplus C_{30}, \qquad C_{30} \oplus C_{300}, \qquad C_{60} \oplus C_{150}.$$

188. Suppose that the linear transformation T acts on the 8 dimension vector space $\mathbb{C}$ over the complex numbers. Use T to make V into a $\mathbb{C}[t]$-module (where t is an indeterminate) in the usual way. Suppose that as a $\mathbb{C}[t]$-module

$$V \cong \frac{\mathbb{C}[t]}{(t-5)^3(t+2)} \oplus \frac{\mathbb{C}[t]}{(t-5)^2(t+2)^2}.$$

(i) What is the Jordan normal form of T.

(ii) What are the eigenvalues of T and how many eigenvectors does T have (up to scalar multiples)?

(iii) What is the minimum polynomial of T?

189. Determine the invariant factors and the torsion free rank of the abelian group M generated by x, y, z subject to the relations

$$9x + 12y + 6z = 0 \qquad \text{and} \qquad 6x + 3y - 6z = 0.$$

190. Let M be the abelian group generated by x, y, z subject to the relations

$$9x + 12y + 6z = 0 \qquad \text{and} \qquad 6x + 3y - 6z = 0.$$

Express M as a direct sum of cyclic groups in a unique way and find the primary components of M.

191. Determine the invariant factors of the abelian group generated by a, b, c, d subject to the relations $3a - 3c = 6b + 3c - 6d = 3b + 2c - 3d = 3a + 6c + 3d = 0$.

192. Give a list of all the different abelian groups of order 54.

193. Consider the linear transformation α acting on $\mathbb{Z}^3$ given by

$$\alpha(e_1) = e_2 + e_3, \qquad \alpha(e_2) = 2e_2, \quad \alpha(e_3) = e_1 + 2e_2.$$

Show that the minimal polynomial and the characteristic polynomial for α are the same.

194. Consider the linear transformation α acting on $\mathbb{F}_3^3$ given by

$$\alpha(e_1) = e_2 + e_3, \qquad \alpha(e_2) = 2e_2, \quad \alpha(e_3) = e_1 + 2e_2.$$

Determine the Jordan normal form of α.

195. Let $p \in \mathbb{Z}_{>0}$ be prime. Show that $\mathbb{Z}/p^2\mathbb{Z}$ is not isomorphic to the direct sum of two cyclic groups.

196. Let A be the abelain group generated by a, b, c with relations

$$7a + 4b + c = 8a + 5b + 2c = 9a + 6b + 3c = 0.$$

Express this group as a direct sum of cyclic groups.

197. If an abelian group has torsion invariants (or equivalently invariant factors) $2, 6, 54$ determine its primary decomposition.

198. Determine all abelian groups of order 72.

199. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & 0 \\ 0 & -1 & 5 \end{pmatrix}$. Show that the minimal polynomial of A is $f(x) = (x-2)^2$ and the characteristic polynomial is $g(x) = (x-2)^3$.

200. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & 0 \\ 0 & -1 & 5 \end{pmatrix}$ and let $V = \mathbb{Q}^3$ be the corresponding $\mathbb{Q}[x]$-module. Prove that

$$V \cong \frac{\mathbb{Q}[x]}{(x-2)} \oplus \frac{\mathbb{Q}[x]}{(x-2)^3} \oplus \frac{\mathbb{Q}[x]}{(x-2)^2}.$$

201. Use the structure theorem for modules to show that a torsion free finitely generated module over a PID is free.

202. Let V be the $\mathbb{Q}[x]$-module with presentation matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & x & 0 & 0 \\ 1 & 0 & 1-x & 1 \\ 0 & 0 & 0 & x^2 \end{pmatrix}.$$

Show that

$$V \cong \frac{\mathbb{Q}[x]}{x\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{x^3\mathbb{Q}[x]}.$$

203. Show (without using the Structure Theorem) that $\mathbb{Z}_{p^r}$ is not a direct sum of two abelian groups when p is a prime and r is a positive integer.

204. Using the Structure theorem or otherwise show that a fintiely generate torsion-free abelian group is a free abelian group.

205. Determine the torsion-free rank and the torsion invariants of the abelian group given by generators a, b, c and relations

$$7a + 4b + c = 8a + 5b + 2c = 9a + 6b + 3c = 0.$$

206. Determine, up to isomorphism, all abelian groups of order 1080.

207. Show that $\mathbb{Q}$ is torsion-free but not free as a $\mathbb{Z}$-module.

208. Show that a finitely generated torsion free module over a Principal Ideal Domain R is a free R-module.

209. Deteremine the torsion free rank and the torsion invariants of the abelian group presented by generators a, b, c and relations

$$7a + 4b + c = 8a + 5b + 2c = 9a + 6b + 3c = 0.$$

210. Determine, up to isomorphism, all abelian groups of order 360.