1.16 Lecture 14: Fields and Integral Domains

1.16.1 R/M is a field $\iff$ M is a maximal ideal.

Definition.

- A **field** is a commutative ring F such that if $x \in F$ and $x \neq 0$ then there exists an element $x^{-1} \in F$ such that $xx^{-1} = 1$.
- A **maximal ideal** is an ideal M of a ring R such that
  (a) $M \neq R$,
  (b) If N is an ideal of R and $M \subsetneq P$ then $P = R$.

Lemma 1.69. *Let F be a commutative ring. Then F is a field if and only if the only ideals of F are $\{0\}$ and F.*

Theorem 1.70. *Let R be a commutative ring and let M be an ideal of R. Then*

$$R/M \text{ is a field} \quad \text{if and only if} \quad M \text{ is a maximal ideal.}$$

1.16.2 R/P is an integral domain $\iff$ P is a prime ideal.

Definition.

- An **integral domain** is a commutative ring R such that

  (Cancellation law) if $a, b, c \in R$ and $c \neq 0$ and $ac = bc$ then $a = b$.
- A **prime ideal** is an ideal P in a commutative ring R such that if $a, b \in R$ and $ab \in P$ then either $a \in P$ or $b \in P$.

The following proposition says that a commutative ring satisfies the cancellation law if and only if R has no zero divisors except 0.

Proposition 1.71. *Let R be a commutative ring. Then R satsifies*

$$\text{If } a, b, c \in R \text{ and } c \neq 0 \text{ and } ac = bc \text{ then } a = b,$$

if and only if R satisfies

$$\text{if } a, b \in R \text{ and } ab = 0 \text{ then either } a = 0 \text{ or } b = 0.$$

Theorem 1.72. *Let R be a commutative ring and let P be an ideal of R. Then*

$$R/P \text{ is an integral domain} \quad \text{if and only if} \quad P \text{ is a prime ideal.}$$

HW:. Show that every field is an integral domain.

HW:. Show that every maximal ideal is prime.

HW:. So that the ideal $x\mathbb{Z}[x]$ in $\mathbb{Z}[x]$ is a prime ideal that is not maximal.