6.3 Polynomial Rings

Definition. Let $\mathbb{A}$ be a commutative ring and for $i \in \mathbb{Z}_{>0}$ let x^i be a formal symbol.

- A **polynomial with coefficients in** $\mathbb{A}$ is an expression of the form

$$a(x) = a_0 + r_1 x + a_2 x^2 + \cdots$$

  such that

  (a) if $i \in \mathbb{Z}_{\geq 0}$ then $a_i \in \mathbb{A}$,

  (b) There exists $N \in \mathbb{Z}_{>0}$ such that if $i \in \mathbb{Z}_{>N}$ then $a_i = 0$.

- Polynomials $f(x) = r_0 + r_1 x + r_2 x^2 + \cdots$ and $g(x) = s_0 + s_1 x + s_2 x^2 + \cdots$ with coefficients in R are

  equal if $\quad r_i = s_i$ for $i \in \mathbb{Z}_{\geq 0}$.

- The **zero polynomial** is the polynomial $0 = 0 + 0x + 0x^2 + \cdots$.

- The **degree** $\deg\big(f(x)\big)$ of a polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ with coefficients in $\mathbb{A}$ is

  the smallest $N \in \mathbb{Z}_{\geq 0}$ such that $\quad a_N \neq 0$ and $a_k = 0$ for $k \in \mathbb{Z}_{>N}$.

  If $f(x) = 0 + 0x + 0x^2 + \cdots$ then define $\deg\big(f(x)\big) = 0$.

- Let $\mathbb{A}$ be a commutative ring. The **ring of polynomials with coefficients in** $\mathbb{A}$ is the set $\mathbb{A}[x]$ of polynomials with coefficients in $\mathbb{A}$ with the operations of addition and multiplication defined as follows:
  If $f(x), g(x) \in \mathbb{A}[x]$ with

$$f(x) = r_0 + r_1 x + r_2 x^2 + \cdots \quad \text{and} \quad g(x) = s_0 + s_1 x + s_2 x^2 + \cdots,$$

  then

$$f(x) + g(x) = (r_0 + s_0) + (r_1 + s_1)x + (r_2 + s_2)x^2 + \cdots, \quad \text{and}$$
$$f(x)g(x) = c_0 + c_1 x + c_2 x^2 + \cdots, \quad \text{where} \quad c_k = \sum_{i+j=k} r_i s_j.$$

Proposition 6.9.

(a) Let R, S be commutative rings and let $\varphi\colon R \to S$ be a ring homomorphism. Then the function

$$\begin{array}{rcl} \psi\colon R[x] & \longrightarrow & S[x] \\ r_0 + r_1 x + r_2 x^2 + \cdots & \longmapsto & \varphi(r_0) + \varphi(r_1)x + \varphi(r_2)x^2 + \cdots \end{array}$$

is a ring homomorphism.

(b) Let R be a commutative ring and let $\alpha \in R$. Then the evaluation homomorphism

$$\begin{array}{rccc} \mathrm{ev}_\alpha\colon & R[x] & \to & R \\ & r_0 + r_1 x + \cdots r_d x^d & \mapsto & r_0 + r_1\alpha + \cdots + r_d \alpha^d \end{array} \quad \textit{is a ring homomorphism.}$$