2.17 Proof that maximal ideals produce fields

Theorem 2.22. *Let R be a commutative ring and let M be an ideal of R. Then R/M is a field if and only if M is a maximal ideal.*

Proof.
$\Rightarrow$: Assume R/M is a field.
Then, by Lemma 4.44, the only ideals of R/M are (0) and R/M.
By the correspondence theorem, Ex. 2.1.5(c), there is a one-to-one correspondence between ideals of R/M and ideals of R containing M.
Thus the only ideals of R containing M are M and R.
So M is a maximal ideal.

$\Leftarrow$: Assume M is a maximal ideal.
Then the only ideals of R containing M are M and R.
By the correspondence theorem, Ex. 2.1.5(c), there is a one-to-one correspondence between ideals of R/M and ideals of R containing M.
Thus the only ideals of R/M are (0) and R/M.
So, by Lemma 4.44, R/M is a field. □

2.18 Proof that the cancellation law is equivalent to the no zero divisor property

Proposition 2.23. *Let R be a commutative ring. Then R satsifies*

$$\text{If } a,b,c \in R \text{ and } c \neq 0 \text{ and } ac = bc \text{ then } a = b,$$

if and only if R satisfies

$$\text{if } a,b \in R \text{ and } ab = 0 \text{ then either } a = 0 \text{ or } b = 0.$$

Proof. $\Rightarrow$: Assume that R has no zero divisors.
Assume $a,b,c \in R$ and $c \neq 0$ and $ac = bc$.
Then $0 = ac - bc = (a-b)c$.
Since R is an integral domain and $c \neq 0$ then $a - b = 0$.
So $a = b$. So R satisfies the cancellation law.

$\Leftarrow$: Assume that R satisfies the cancellation law.
To show: If $a,b \in R$ and $ab = 0$ then either $a = 0$ or $b = 0$.
Assume $a,b \in R$ and $ab = 0$.
To show: Either $a = 0$ or $b = 0$.
To show: If $a \neq 0$ then $b = 0$.
Assume $a \neq 0$.
To show: $b = 0$.
Then $ab = 0 = a \cdot 0$.
Since $a \neq 0$ then the cancellation law gives $b = 0$. □