<u>The ring of integers of $\mathbb{Q}(\sqrt{d})$</u>

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Algebra Lect. 27
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Assume $d \in \mathbb{Z}$ and $\sqrt{d} \notin \mathbb{Z}$.

The <u>ring of integers of $\mathbb{Q}(\sqrt{d})$</u> is

<u>Case 1:</u> $d \notin 1+4\mathbb{Z}$

$$\begin{aligned}\mathbb{Z}[\sqrt{d}] &= \mathbb{Z}\text{-span}\{1, \sqrt{d}\} \\ &= \{a+b\sqrt{d} \mid a, b \in \mathbb{Z}\}.\end{aligned}$$

<u>Case 2:</u> $d \in 1+4\mathbb{Z}$

$$\begin{aligned}\mathbb{Z}[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{d}] &= \mathbb{Z}\text{-span}\{1, \tfrac{1}{2}+\tfrac{1}{2}\sqrt{d}\} \\ &= \left\{\frac{a+b\sqrt{d}}{2} \;\middle|\; \begin{array}{l} a, b \in \mathbb{Z} \text{ and} \\ a \text{ and } b \text{ are both even} \\ \text{or } a \text{ and } b \text{ are both odd} \end{array}\right\}\end{aligned}$$

~~All of~~ In case 2,

$$\mathbb{Z}[\sqrt{d}] = \left\{\frac{a+b\sqrt{d}}{2} \;\middle|\; \begin{array}{l} a, b \in \mathbb{Z} \text{ and} \\ a \text{ and } b \text{ are both even} \end{array}\right\}$$

is a subring of $\mathbb{Z}[\frac{1}{2}+\frac{1}{2}\sqrt{d}]$.

All of these are subrings of $\mathbb{C}$.

Let R be the ring of integers of $\mathbb{Q}(\sqrt{d})$.

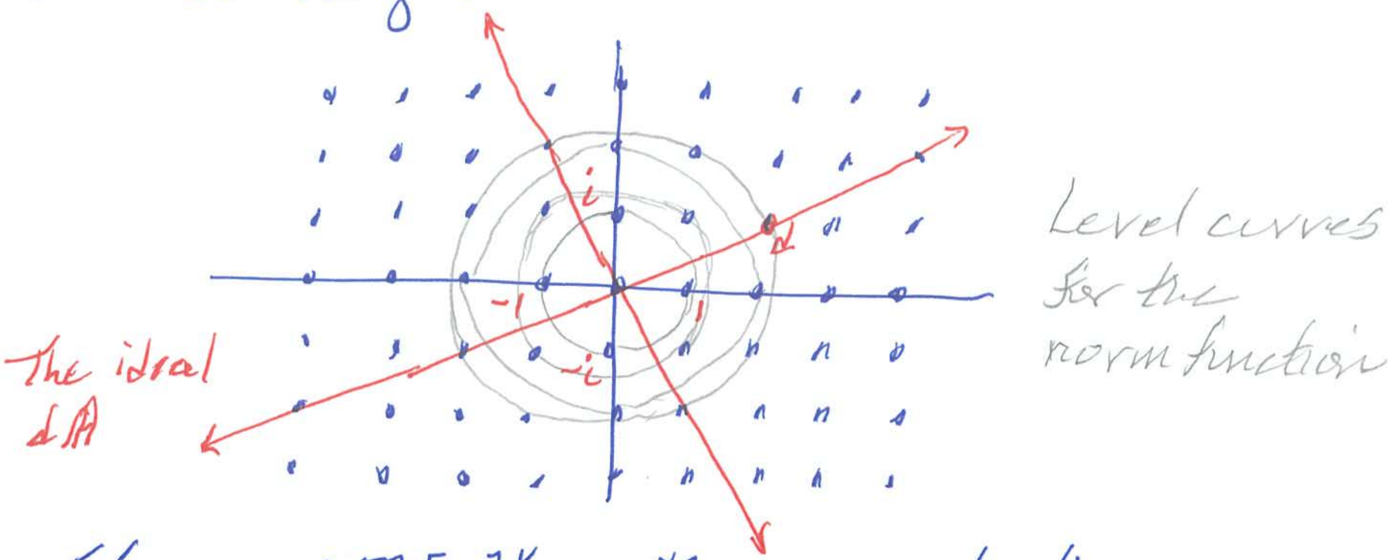
The <u>norm function</u> on R is

$$\begin{aligned} N: R &\longrightarrow \mathbb{Z}_{\geq 0} \\ z &\longmapsto |z|^2 = z\bar{z}. \end{aligned}$$

The Gaussian integers

$$\mathbb{Z}[i] = \mathbb{Z}[\sqrt{-1}] = \{a+bi \mid a, b \in \mathbb{Z}\}$$

is a subring of $\mathbb{C}$



If $z, u \in \mathbb{Z}[i]^\times$ with $zu = 1$ then

$$N(z)N(u) = N(zu) = N(1) = 1$$

so that $N(z) = 1$. So

$$\mathbb{Z}[i]^\times = \{1, i, -1, i\}$$

Proposition

(a) $(\mathbb{Z}[i], N)$ is a Euclidean domain

(b) $\mathbb{Z}[i]$ is a PID and a UFD.

$$\text{first quadrant} \longleftrightarrow \mathbb{Z}[i]/\mathbb{Z}[i]^\times \longleftrightarrow \left\{\begin{matrix}\text{ideals} \\ \text{of } \mathbb{Z}[i]\end{matrix}\right\}$$

$$\alpha \longmapsto \alpha\mathbb{Z}[i]^\times \longmapsto \alpha A, \quad \text{where } A = \mathbb{Z}[i]$$

αA is a 4-spoked laser ray emanating from 0 with distances α.

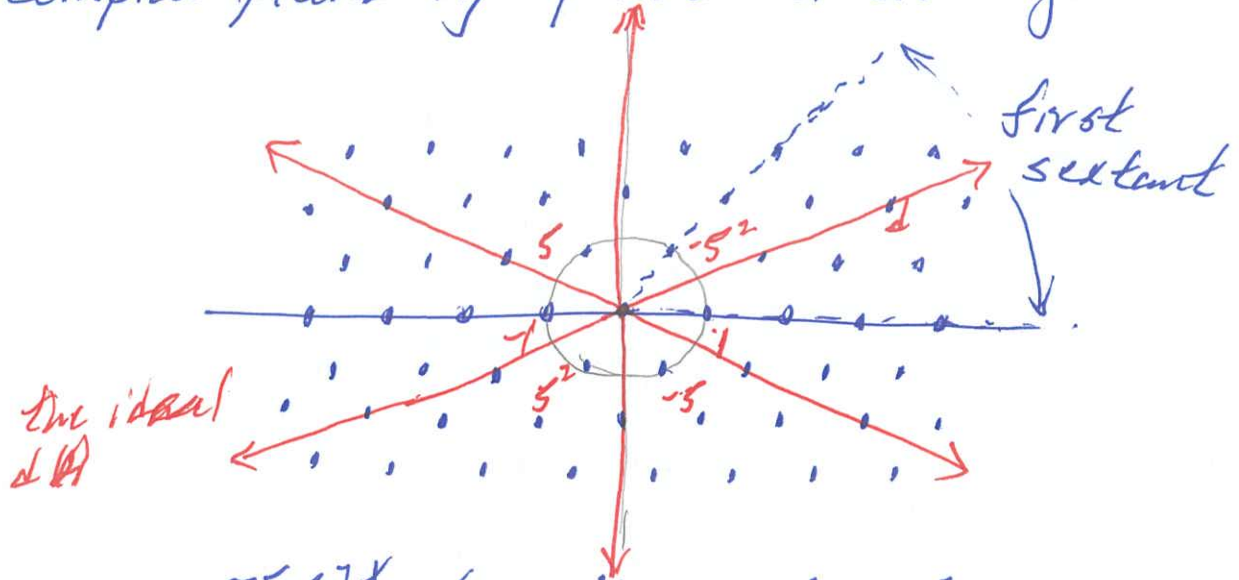
Which are the maximal ideals?

The Eisenstein integers $\mathbb{Z}[\zeta]$

Let $\zeta = e^{2\pi i/3} = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

$$\mathbb{Z}[\zeta] = \mathbb{Z}\text{-span}\{1, \zeta\} = \{a + b\zeta \mid a, b \in \mathbb{Z}\}$$

The points of $\mathbb{Z}[\zeta]$ produce a tiling of the complex plane by equilateral triangles.



$$\mathbb{Z}[\zeta]^\times = \{1, -\zeta^2, \zeta, -1, \zeta^2, -\zeta\}$$

Proposition

(a) $(\mathbb{Z}[\zeta], N)$ is a Euclidean domain

(b) $\mathbb{Z}[\zeta]$ is a PID and a UFD

$$\begin{array}{ccccc} \text{first sextant} & \longleftrightarrow & \mathbb{Z}[\zeta]/\mathbb{Z}[\zeta]^\times & \longleftrightarrow & \{\text{ideals of } \mathbb{Z}[\zeta]\} \\ d & \longmapsto & d\,\mathbb{Z}[\zeta]^\times & \longmapsto & d\,\mathbb{Z}[\zeta] \end{array}$$

$d\mathbb{Z}[\zeta]$ is a 6-spoked laser ray emanating from 0 with distances d.

Which are the maximal ideals?

The ring $\mathbb{Z}[\sqrt{-5}]$ is not a UFD, for example,

$$3 \cdot 3 = 9 = (2+\sqrt{-5})(2-\sqrt{-5}) \quad \text{and}$$
$$2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$$

So $\mathbb{Z}[+\sqrt{5}]$ is not a PID and not a Euclidean domain for any size function.

<u>Theorem</u> (lots of hard work from many authors combined)

Let $d \in \mathbb{Z}_{>0}$ and let $\mathbb{A}(\sqrt{-d})$ be the ring of integers of $\mathbb{Q}(\sqrt{-d})$.

(a) $(\mathbb{A}(\sqrt{-d}), N)$ is a Euclidean domain if and only if

$$d \in \{1, 2, 3, 7, 11\}$$

(b) $\mathbb{A}(\sqrt{-d})$ is a PID if and only if

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$$

(c) $\mathbb{A}(\sqrt{-d})$ is a UFD if and only if

$$d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$$

See the book of Shult and Surowski; Proposition 9.A.1.