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Algebra Lect 25
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(1)

$\Lambda^k(V)$ and $\Lambda^k(T)$

Let $\mathbb{F}$ be a field and $k, n \in \mathbb{Z}_{>0}$.
Let V be an $\mathbb{F}$-vector space with $\dim(V) = n$
and $T: V \to V$ an $\mathbb{F}$-linear transformation.

The vector space $\Lambda^k(V)$ is generated by symbols

$$v_1 \wedge \cdots \wedge v_k \quad \text{with } v_1, \ldots, v_k \in V$$

and relations

$$\begin{aligned} v_1 \wedge \cdots \wedge v_{i-1} \wedge (c_1 v + c_2 w) \wedge v_{i+1} \wedge \cdots \wedge v_k &= c_1 v_1 \wedge \cdots \wedge v_{i-1} \wedge v \wedge v_{i+1} \wedge \cdots \wedge v_k \\ &\quad + c_2 v_1 \wedge \cdots \wedge v_{i-1} \wedge w \wedge v_{i+1} \wedge \cdots \wedge v_k \end{aligned}$$

and

$$\begin{aligned} v_1 \wedge \cdots \wedge v_{j-1} \wedge v_j \wedge v_{j+1} \wedge v_{j+2} \wedge \cdots \wedge v_k \\ = - v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge v_j \wedge v_{j+2} \wedge \cdots \wedge v_k \end{aligned}$$

for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, k-1\}$, $v, w \in V$ and $c_1, c_2 \in \mathbb{F}$.

The <u>linear transformation</u> $\Lambda^k(T): \Lambda^k(V) \to \Lambda^k(V)$ is given by

$$\Lambda^k(T)(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k).$$

Example Suppose $\mathbb{Q}$-span$\{e_1, e_2, e_3\} = V$ so that $\dim_{\mathbb{Q}}(V) = 3$ and $\{e_1, e_2, e_3\}$ is a basis of V.

If $v_1 = 3e_1 + 2e_2 + e_3$ and $v_2 = 4e_1 + 5e_3$

then
$$v_1 \wedge v_2 = (3e_1 + 2e_2 + e_3) \wedge (4e_1 + 5e_3)$$
$$= 3\, e_1 \wedge (4e_1 + 5e_3) + 2 e_2 \wedge (4e_1 + 5e_3) + e_3 \wedge (4e_1 + 5e_3)$$
$$= 3\cdot 4\, e_1 \wedge e_1 + 3\cdot 5\, e_1 \wedge e_3 + 2\cdot 4\, e_2 \wedge e_1 + 2\cdot 5\, e_2 \wedge e_3 + 4 e_3 \wedge e_1 + 5 e_3 \wedge e_3$$
$$= 3\cdot 4\cdot 0 + 3\cdot 5\, e_1 \wedge e_3 - 2\cdot 4\, e_1 \wedge e_2 + 2\cdot 5\, e_2 \wedge e_3 - 4 e_1 \wedge e_3 + 5\cdot 0$$

since $e_1 \wedge e_1 = -e_1 \wedge e_1$ gives $e_1 \wedge e_1 = 0$
and $e_3 \wedge e_3 = -e_3 \wedge e_3$ gives $e_3 \wedge e_3 = 0$.

So $v_1 \wedge v_2 = -8 e_1 \wedge e_2 + 9 e_1 \wedge e_3 + 10 e_2 \wedge e_3$.

Proposition If V has $\mathbb{F}$-basis $\{e_1, \ldots, e_n\}$ then $\Lambda^k V$ has $\mathbb{F}$-basis
$$\left\{ e_{i_1} \wedge \cdots \wedge e_{i_k} \;\middle|\; \begin{array}{l} i_1, \ldots, i_k \in \{1, \ldots, n\} \\ \text{and } i_1 < i_2 < \cdots < i_k \end{array} \right\}$$

Example: Suppose V has $\mathbb{Q}$-basis $\{e_1, e_2, e_3\}$ and $T: V \to V$ is the linear transformation with matrix

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & -1 \\ 3 & 0 & 4 \end{pmatrix}$$ with respect to the basis $\{e_1, e_2, e_3\}$.

Then $\Lambda^2 V$ has $\mathbb{Q}$-basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and $\Lambda^2(T): \Lambda^2(V) \to \Lambda^2(V)$ has matrix

$$\Lambda^2(A) = \begin{pmatrix} 1\cdot 5 - 2\cdot 4 & 1\cdot(-1) - 2\cdot 3 & 4\cdot(-1) - 3\cdot 5 \\ 1\cdot 0 - 3\cdot 4 & 1\cdot 4 - 3\cdot 3 & 4\cdot 4 - 0\cdot 3 \\ 2\cdot 0 - 3\cdot 5 & 2\cdot 4 - 3(-1) & 5\cdot 4 - 0\cdot(-1) \end{pmatrix}$$

with respect to the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$.

For example:

$$\Lambda^2(T)(e_1 \wedge e_2) = Te_1 \wedge Te_2 = (e_1 + 2e_2 + 3e_3) \wedge (4e_1 + 5e_2)$$

$$= 1\cdot 4\, e_1 \wedge e_1 + 1\cdot 5\, e_1 \wedge e_2 + 2\cdot 4\, e_2 \wedge e_1 + 2\cdot 5\, e_2 \wedge e_2 + 3\cdot 4\, e_3 \wedge e_1 + 3\cdot 5\, e_3 \wedge e_2$$

$$= 0 + (1\cdot 5 - 2\cdot 4)\, e_1 \wedge e_2 + 0 + (-3\cdot 4)\, e_1 \wedge e_3 + -3\cdot 5\, e_2 \wedge e_3$$

which verifies the entries in the first column of $\Lambda^2(A)$.

<u>Proposition</u> Let V be a vector space with basis A. Ra $\{e_1, \dots, e_n\}$. If $T: V \to V$ is a linear transformation with matrix $A = (a_{ij})$ with respect to the basis $\{e_1, \dots, e_n\}$ then

with respect to the basis $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \le i_1 < \dots < i_k \le n$

the linear transformation

$\Lambda^k(T): \Lambda^k V \to \Lambda^k V$ has matrix $\Lambda^k(A)$

where the $((i_1, \dots, i_k), (j_1, \dots, j_k))$ entry of $\Lambda^k(A)$ is

$$\det \begin{pmatrix} a_{i_1 j_1} & \cdots & a_{i_k j_1} \\ \vdots & & \vdots \\ a_{i_k j_1} & \cdots & a_{i_k j_k} \end{pmatrix}$$

Recall the Smith Normal Form:

<u>Theorem</u> Let $\mathbb{A}$ be a PID and $A \in M_{t \times s}(\mathbb{A})$.
~~Let~~ Then there exist $P \in GL_t(\mathbb{A})$ and $Q \in GL_s(\mathbb{A})$

and $d_1, \dots, d_k \in \mathbb{A}$ such that $k = \min(s, t)$

$$d_1 \mathbb{A} \supseteq d_2 \mathbb{A} \supseteq \dots \supseteq d_k \mathbb{A}$$

and $A = PDQ$ with $D = \operatorname{diag}(d_1, \dots, d_k)$

Theorem If $\ell \in \{1, \ldots, k\}$ then

$$d_1 \cdots d_\ell = \gcd(\text{entries of } \Lambda^\ell(A)).$$

Proof idea: Multiplying a matrix A by an invertible matrix P or Q does not change the gcd of its entries. Since

$$\Lambda^\ell(A) = \Lambda^\ell(PDQ) = \Lambda^\ell(P)\Lambda^\ell(D)\Lambda^\ell(Q)$$

then

$$\gcd\begin{pmatrix}\text{entries of} \\ \Lambda^\ell(A)\end{pmatrix} = \gcd\begin{pmatrix}\text{entries} \\ \text{of } \Lambda^\ell(D)\end{pmatrix} = d_1 \cdots d_\ell. \quad //$$

Example $\mathbb{A} = \mathbb{Z}$ and

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & -1 \\ 3 & 0 & 4 \end{pmatrix} = \Lambda^1(A), \qquad \Lambda^2(A) = \begin{pmatrix} -3 & -7 & -19 \\ -12 & -5 & 16 \\ -15 & 11 & 20 \end{pmatrix}$$

$$\Lambda^3(A) = (\det(A)) = (-23 \cdot 3).$$

So $A = PDQ$ with $D = \mathrm{diag}(d_1, d_2, d_3)$ where

$$d_1 = 1, \quad d_1 d_2 = 1, \quad d_3 d_2 d_1 = -23 \cdot 3.$$

So $d_1 = 1$, $d_2 = 1$, $d_3 = -23 \cdot 3$ and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -23 \cdot 3 \end{pmatrix}.$$