Principal ideals

Let R be a commutative ring.
Let $p \in R$ with $p \neq 0$ and $p \notin R^\times$.

The element p is prime if p satisfies:
if p divides ab then
p divides a or p divides b.

In other words:
If $ab \in pR$ then $a \in pR$ or $b \in pR$.

In other words:
p is prime if pR is a prime ideal.

The element p is irreducible if
there do not exist $a, b \in R$
such that $a, b \notin R^\times$ and $p = ab$.

In other words: There does not exist
$a \in R$ such that $pR \subsetneq aR \subsetneq R$.

In other words: p is irreducible if
pR is a maximal principal ideal.

Examples

(1) $R = \mathbb{Z}[x]$. The element x is irreducible and prime.

The ideal xR is a maximal principal ideal but xR is not a maximal ideal.

(2) $R = \mathbb{Z}[\sqrt{-5}]$. The element 3 is irreducible but not prime. For example

$$3 \text{ divides } (1+\sqrt{-5})(1-\sqrt{-5}) = 6$$

but 3 does not divide $1+\sqrt{-5}$ and 3 does not divide $1-\sqrt{-5}$.

Let

$$\mathcal{S}_{[0,R]} = \{\text{ideals of } R\}$$

$$\mathcal{P}_{[0,R]} = \{\text{principal ideals of } R\}$$

A <u>principal ideal domain</u>, or <u>PID</u>, is an integral domain such that

$$\mathcal{S}_{[0,R]} = \mathcal{P}_{[0,R]}.$$

Proposition Let R be an integral domain.

(a) $$\begin{matrix} R/R^\times & \longrightarrow & \mathcal{P}_{[0,R]} \\ dR^\times & \longmapsto & dR \end{matrix}$$ is a bijection.

(b) Let $d \in R$ such that $d \neq 0$ and $d \notin R^\times$.

If d is prime then d is irreducible.

(c) Let $d \in R$ such that $d \neq 0$ and $d \notin R^\times$.

If $\mathcal{S}_{[0,R]} = \mathcal{P}_{[0,R]}$ and d is irreducible then d is prime.

(d) Let $d \in R$ such that $d \neq 0$ and $d \notin R^\times$.

If $\mathcal{P}_{[0,R]}$ satisfies ACC then there exist $k \in \mathbb{Z}_{>0}$ and irreducible $p_1, \ldots, p_k \in R$ such that $d = p_1 \cdots p_k$.

(e) If $\mathcal{S}_{[0,R]} = \mathcal{P}_{[0,R]}$

then $\mathcal{P}_{[0,R]}$ satisfies ACC.

Proof of (d)

Assume $d \in R$ and $d \neq 0$ and $d \notin R^\times$.

To show: If d does not have a finite factorization into irreducible elements then $\mathcal{P}_{[0,R]}$ does not satisfy ACC.

Assume d does not have a finite factorization into irreducible elements.

Then d is not irreducible and there exists $d_1 \in R$ such that

$$dR \subsetneq d_1R \subsetneq R.$$

Then d_1 is not irreducible and there exists $d_2 \in R$ such that

$$dR \subsetneq d_1R \subsetneq d_2R \subsetneq R$$

Continuing this process produces an infinite chain

$$dR \subsetneq d_1R \subsetneq d_2R \subsetneq \cdots$$

of principal ideals in R.

So $\mathcal{P}_{[0,R]}$ does not satisfy ACC.