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Algebra Lect. 6 ①
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Möbius transformations

$\mathbb{C}[t] = \{\text{polynomials in } t \text{ with coefficients in } \mathbb{C}\}$

$= \{a_0 + a_1 t + \cdots + a_\ell t^\ell \mid \ell \in \mathbb{Z}_{\geq 0}, a_0, \ldots, a_\ell \in \mathbb{C}\}$

$\mathbb{C}(t) = \left\{ \frac{a(t)}{b(t)} \mid a(t), b(t) \in \mathbb{C}[t] \text{ and } b(t) \neq 0 \right\}$

with

$$\frac{a(t)}{b(t)} = \frac{c(t)}{d(t)} \quad \text{if } a(t)d(t) = b(t)c(t).$$

The group of units of $M_{2\times 2}(\mathbb{C})$ is

$$GL_2(\mathbb{C}) = M_{2\times 2}(\mathbb{C})^\times = \left\{ A \in M_{2\times 2}(\mathbb{C}) \;\middle|\; \begin{array}{l} \text{there exists} \\ B \in M_{2\times 2}(\mathbb{C}) \text{ with} \\ AB = BA = 1 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{C}) \;\middle|\; ad - bc \neq 0 \right\}$$

<u>Theorem</u> The map

$$\begin{array}{ccc} GL_2(\mathbb{C}) & \longrightarrow & \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(t)) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \sigma_{\substack{ab \\ cd}} \end{array},$$

where $\sigma_{\substack{ab \\ cd}} : \mathbb{C}(t) \to \mathbb{C}(t)$

$$t \longmapsto \frac{at+b}{ct+d},$$

is a group homomorphism.

If $\frac{f(t)}{g(t)} \in \mathbb{C}(t)$ then

$$\sigma_{\substack{ab\\cd}}\left(\frac{f(t)}{g(t)}\right) = \frac{f\left(\frac{at+b}{ct+d}\right)}{g\left(\frac{at+b}{ct+d}\right)}$$

<u>Example</u> Let $\mathbb{K} = \mathbb{C}(t)$ and

$\alpha = t^4 + t^{-4}$.

$$\begin{array}{c} \mathbb{C}(t) = \mathbb{K} \\ \cup\!| \\ \mathbb{C}(\alpha) = \mathbb{F} \end{array}$$

Let $\sigma, \tau \in \mathrm{Aut}_{\mathbb{F}}(\mathbb{C}(t))$ be

$$\sigma = \sigma_{\substack{i0\\01}} \quad \text{and} \quad \tau = \sigma_{\substack{01\\10}}$$

Since

$$\sigma(\alpha) = \sigma(t^4 + t^{-4}) = \left(\frac{it+0}{0t+1}\right)^4 + \frac{1}{\left(\frac{it+0}{0t+1}\right)^4}$$

$$= (it)^4 + \frac{1}{(it)^4} = t^4 + t^{-4} = \alpha$$

and

$$\tau(t) = \tau(t^4 + t^{-4}) = \left(\frac{0t+1}{1t+0}\right)^4 + \frac{1}{\left(\frac{0t+1}{1t+0}\right)^4}$$

$$= \left(\frac{1}{t}\right)^4 + \frac{1}{\left(\frac{1}{t}\right)^4} = t^{-4} + t^4 = \alpha$$

then

$$\sigma, \tau \in \mathrm{Aut}_{\mathbb{C}(\alpha)}(\mathbb{C}(t)).$$

The subgroup of $\mathrm{Aut}_{\mathbb{C}(\alpha)}/\mathbb{C}(\varepsilon)$ generated by σ and τ is

$$D_4 = \{1, \sigma, \sigma^2, \sigma^3, \tau, \tau\sigma, \tau\sigma^2, \tau\sigma^3\}$$

with $\sigma^4 = 1$, $\tau^2 = 1$, $\sigma\tau = \tau\sigma^3$.

Let $\mathbb{F} = \mathbb{C}(\alpha)$ and

$$f(x) = x^8 - \alpha x^4 + 1 \quad \text{in } \mathbb{F}[x].$$

Then

$$\{\varepsilon, \sigma\varepsilon, \sigma^2\varepsilon, \sigma^3\varepsilon, \tau\varepsilon, \tau\sigma\varepsilon, \tau\sigma^2\varepsilon, \tau\sigma^3\varepsilon\}$$

$$= \{\varepsilon, i\varepsilon, -\varepsilon, -i\varepsilon, \varepsilon^{-1}, -i\varepsilon^{-1}, -\varepsilon^{-1}, i\varepsilon^{-1}\}$$

are 8 distinct roots of f in $\mathbb{C}(\varepsilon)$.

So $f(x) = (x-\varepsilon)(x+\varepsilon)(x-i\varepsilon)(x+i\varepsilon)$

$$\cdot(x-\varepsilon^{-1})(x+\varepsilon^{-1})(x-i\varepsilon^{-1})(x+i\varepsilon^{-1})$$

$$= m_{\varepsilon,\mathbb{C}(\alpha)}(x)$$

and

$\mathbb{C}(\varepsilon)$ is the splitting field of $f(x)$ over $\mathbb{F}$.

The Galois correspondence

$$\begin{tikzcd}
 & & \{1\} \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\{1,\tau\} \arrow[dr, dash] & \{1,\tau\sigma^2\} \arrow[d, dash] & \{1,\sigma^2\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \{1,\tau\sigma\} \arrow[d, dash] & \{1,\tau\sigma^3\} \arrow[dl, dash] \\
 & \{1,\sigma^2,\tau,\tau\sigma^2\} \arrow[dr, dash] & \{1,\sigma,\sigma^2,\sigma^3\} \arrow[d, dash] & \{1,\sigma^2,\tau\sigma,\tau\sigma^3\} \arrow[dl, dash] & \\
 & & D_4 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \mathbb{C}(\varepsilon) \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & \\
\mathbb{C}(\varepsilon+\varepsilon^{-1}) \arrow[dr, dash] & \mathbb{C}(\varepsilon-\varepsilon^{-1}) \arrow[d, dash] & \mathbb{C}(\varepsilon^2) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \mathbb{C}(\varepsilon-i\varepsilon^{-1}) \arrow[d, dash] & \mathbb{C}(\varepsilon+i\varepsilon^{-1}) \arrow[dl, dash] \\
 & \mathbb{C}(\varepsilon^2+\varepsilon^{-2}) \arrow[dr, dash] & \mathbb{C}(\varepsilon^4) \arrow[d, dash] & \mathbb{C}(\varepsilon^2-\varepsilon^{-2}) \arrow[dl, dash] & \\
 & & \mathbb{C}(\varepsilon^4+\varepsilon^{-4}) & &
\end{tikzcd}$$