1.10 Lecture 10: Simple and indecomposable modules and torsion submodules

1.10.1 The Krull-Schmidt theorem

Theorem 1.29. *Let $\mathbb{A}$ be a PID and let M be a finitely generated $\mathbb{A}$ module. Then there exist $k, \ell \in \mathbb{Z}_{\geq 0}$ and $d_1, \ldots, d_k \in \mathbb{A}$ such that*

$$M \cong \frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}} \oplus \mathbb{A}^{\oplus \ell}$$

Special cases of $\mathbb{A}/d\mathbb{A}$ are

$$\frac{\mathbb{A}}{0\mathbb{A}} = \mathbb{A} \qquad \text{and} \qquad \text{if } u \in \mathbb{A}^\times \text{ then } \quad \frac{\mathbb{A}}{u\mathbb{A}} = \frac{\mathbb{A}}{\mathbb{A}} = 0.$$

Theorem 1.30. *(Chinese remainder theorem) Let $\mathbb{A}$ be a PID and let $d \in \mathbb{A}$.*

$$\textit{Assume } d = pq \textit{ with } \gcd(p, q) = 1. \qquad \textit{Then} \qquad \frac{\mathbb{A}}{d\mathbb{A}} \cong \frac{\mathbb{A}}{p\mathbb{A}} \oplus \frac{\mathbb{A}}{q\mathbb{A}}.$$

Theorem 1.31. *(Krull-Schmidt) Let $\mathbb{A}$ be a PID and let M be a finitely generated $\mathbb{A}$-module. Then there exist $r, \ell \in \mathbb{Z}_{>0}$ and indecomposable $\mathbb{A}$-modules $\mathbb{A}/p_1^{k_1}\mathbb{A}, \ldots, \mathbb{A}/p_\ell^{k_\ell}\mathbb{A}$ such that*

$$M \cong \mathbb{A}^{\oplus r} \oplus \frac{\mathbb{A}}{p_1^{k_1}\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{p_\ell^{k_\ell}\mathbb{A}}.$$

1.10.2 Simple and indecomposable modules

Let R be a ring and let M be an R-module.

- The $\mathbb{A}$-module M is **indecomposable** if

  there do not exist submodules N and P of M such that $M = N \oplus P$.

- The R-module M is **simple** if the only submodules of M are 0 and M.
- A **finite composition series of** M is a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \qquad \text{such that } M_i/M_{i+1} \text{ is simple and } n \in \mathbb{Z}_{>0}.$$

Theorem 1.32. *Let $\mathbb{A}$ be a PID.*

(a) There is a bijection

$$\begin{array}{ccc} \{\textit{simple } \mathbb{A}\textit{-modules}\} & \longleftrightarrow & \{\textit{maximal ideals}\} \\ \frac{\mathbb{A}}{p\mathbb{A}} & \leftarrow\!\shortmid & p\mathbb{A} \end{array}$$

(b) There is a bijection

$$\begin{array}{ccc} \{\textit{indecomposable } \mathbb{A}\textit{-modules}\} & \longleftrightarrow & \{(p\mathbb{A}, k) \mid p\mathbb{A} \textit{ is a maximal ideal and } k \in \mathbb{Z}_{>0}\} \\ \frac{\mathbb{A}}{p^k\mathbb{A}} & \leftarrow\!\shortmid & (p\mathbb{A}, k) \end{array}$$

(c) Let $p\mathbb{A}$ be a maximal ideal of $\mathbb{A}$ and let $k \in \mathbb{Z}_{>0}$. The $\mathbb{A}$-module $\mathbb{A}/p^k\mathbb{A}$ has a unique composition series,

$$\frac{\mathbb{A}}{p^k\mathbb{A}} \supseteq \frac{p\mathbb{A}}{p^k\mathbb{A}} \supseteq \cdots \supseteq \frac{p^{k-1}\mathbb{A}}{p^k\mathbb{A}} \supseteq \frac{p^k\mathbb{A}}{p^k\mathbb{A}} = 0.$$

1.10.3 Free modules and torsion submodules

A **integral domain** is a commutative ring $\mathbb{A}$ such that

(Cancellation law) If $a, b, c \in \mathbb{A}$ and $c \neq 0$ and $ac = bc$ then $a = b$.

Let R be a integral domain and let M be an R-module.

- The **torsion submodule** of M is

$$\mathrm{Tor}(M) = \{m \in M \mid \text{there exists } a \in R \text{ with } a \neq 0 \text{ and } am = 0\}.$$

- The module M is **free of finite rank** if there exists $r \in \mathbb{Z}_{>0}$ such that $M \cong \mathbb{A}^{\oplus r}$.

Proposition 1.33. *Let R be an integral domain and let M be an R-module.*

(a) If M is an R-module then $\mathrm{Tor}(M)$ *is an R-submodule of M.*

(b) If M and N are R-modules then $\mathrm{Tor}(M \oplus N) = \mathrm{Tor}(M) \oplus \mathrm{Tor}(N)$.

(c) $\mathrm{Tor}(R) = 0$.

(d) If $d \in R$ and $d \neq 0$ then $\mathrm{Tor}(R/dR) = R/dR$.

Proposition 1.34. *Let $\mathbb{A}$ be a PID. Assume that M is an $\mathbb{A}$-module and there exist $r, k \in \mathbb{Z}_{>0}$ and $d_1, \ldots, d_k \in (\mathbb{A} - \{0, 1\})/\mathbb{A}^\times$ such that*

$$M \cong \mathbb{A}^{\oplus r} \oplus \Big(\frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}}\Big). \qquad \textit{Then} \qquad \mathrm{Tor}(M) \cong \frac{\mathbb{A}}{d_1\mathbb{A}} \oplus \cdots \oplus \frac{\mathbb{A}}{d_k\mathbb{A}}.$$

Proposition 1.35. *Let $\mathbb{A}$ be a PID. Let M be an $\mathbb{A}$-module and let N be an $\mathbb{A}$-submodule of M.*

If M is free of finite rank then N is free of finite rank.