19.6.2 $\mathbb{F}[x]$-Modules

69. Let $A \in M_7(\mathbb{C})$ and suppose that the invariant factors of the matrix $xI - A \in M_7(\mathbb{C}[x])$ are $1, 1, 1, 1, x, x(x-i), x(x-i)^3$.

    (a) Give the corresponding decomposition of $\mathbb{C}^7$ regarded as a $\mathbb{C}[x]$-module.

    (b) Give the Jordan normal form of the matrix A.

    (c) Give the minimal and characteristic polynomials of A.

    (d) Is A diagonalizable?

70. Let V be the $\mathbb{Q}[x]$-module with presentation matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & x & 0 & 0 \\ 1 & 0 & 1-x & 1 \\ 0 & 0 & 0 & x^2 \end{pmatrix}.$$

    Show that

$$V \cong \frac{\mathbb{Q}[x]}{x\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{x^3\mathbb{Q}[x]}.$$

71. Calculate the invariant factor matrix over $\mathbb{Q}[x]$ for the matrix

$$\begin{pmatrix} 1 & x & -2 \\ x+4 & -3 & -6 \\ 2 & -2 & x-3 \end{pmatrix}$$

72. Let V be an 8 dimensional complex vector space and $T\colon V \to V$ a linear transformation.

    (i) Explain how T can be used to define a $\mathbb{C}[x]$-module structure on V.

    (ii) Suppose that as a $\mathbb{C}[x]$ module

$$V \cong \frac{\mathbb{C}[x]}{(x-2)^2(x+3)^2} \oplus \frac{\mathbb{C}[x]}{(x-2)(x+3)^3}.$$

    What is the Jordan normal form for the transformation T? What is the minimal polynomial of T?

73. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & 0 \\ 0 & -1 & 5 \end{pmatrix}$. Show that the minimal polynomial of A is $f(x) = (x-2)^2$ and the characteristic polynomial is $g(x) = (x-2)^3$.

74. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & 0 \\ 0 & -1 & 5 \end{pmatrix}$ and let $V = \mathbb{Q}^3$ be the corresponding $\mathbb{Q}[x]$-module. Prove that

$$V \cong \frac{\mathbb{Q}[x]}{(x-2)} \oplus \frac{\mathbb{Q}[x]}{(x-2)^3} \oplus \frac{\mathbb{Q}[x]}{(x-2)^2}.$$

75. Suppose that the linear transformation T acts on the 8 dimension vector space $\mathbb{C}$ over the complex numbers. Use T to make V into a $\mathbb{C}[t]$-module (where t is an indeterminate) in the usual way. Suppose that as a $\mathbb{C}[t]$-module

$$V \cong \frac{\mathbb{C}[t]}{(t-5)^3(t+2)} \oplus \frac{\mathbb{C}[t]}{(t-5)^2(t+2)^2}.$$

(i) What is the Jordan normal form of T.

(ii) What are the eigenvalues of T and how many eigenvectors does T have (up to scalar multiples)?

(iii) What is the minimum polynomial of T?

76. Let $R = \mathbb{Q}[x]$ and suppose that the torsion R-module M is a direct sum of four cyclic modules whose annihilators (order ideals) are

$$(x-1)^3, \qquad (x^2+1)^2, \qquad (x-1)(x^2+1)^4 \qquad \text{and} \qquad (x+2)(x^2+1)^2.$$

Determine the primary components and invariant factors of M.

77. Let $R = \mathbb{Q}[x]$ and suppose that the torsion R-module M is a direct sum of four cyclic modules whose annihilators (order ideals) are

$$(x-1)^3, \qquad (x^2+1)^2, \qquad (x-1)(x^2+1)^4 \qquad \text{and} \qquad (x+2)(x^2+1)^2.$$

If M is thought of as a vector space over $\mathbb{Q}$ on which x acts as a linear transformation denoted A, determine the minimum and characteristic polynomials of A and the dimension of M over $\mathbb{Q}$.

78. Let $R = \mathbb{C}[x]$ and suppose that the torsion R-module M is a direct sum of four cyclic modules whose annihilators (order ideals) are

$$(x-1)^3, \qquad (x^2+1)^2, \qquad (x-1)(x^2+1)^4 \qquad \text{and} \qquad (x+2)(x^2+1)^2.$$

If M is thought of as a vector space over $\mathbb{C}$ on which x acts as a linear transformation denoted A then is A diagonalizable?

79. Llet T be a linear operator on the finite dimensional vector space V over $\mathbb{C}$. Suppose that the characteristic polynomial of T is $(t+2)^2(t-5)^3$. Determine all possible Jordan forms for a matrix of T. In each case find the minimal polynomial for T and the dimension of the space of eigenvectors.

80. Let V be an eight dimensional complex vector space and let $T\colon V \to V$ be a linear transformation. Explain how V can be regarded as a $\mathbb{C}[t]$-module.

81. Let V be an eight dimensional complex vector space and let $T\colon V \to V$ be a linear transformation. Suppose that

$$V \cong \frac{\mathbb{C}[t]}{(t-2)(1-3)^2} \oplus \frac{\mathbb{C}[t]}{(t-2)(t-3)^3}, \qquad \text{as a } \mathbb{C}[t]\text{-module.}$$

(i) What is the Jordan normal form of T?

(ii) What is the minimal polynomial of T?

(iii) What is the dimension of the eigenspace corresponding to the eigenvalue 3?

82. Let $A \in M_8(\mathbb{C})$ be a matrix and suppose that the matrix $xI - A \in M_8(\mathbb{C}]x])$ is equivalent to the matrix
$$\mathrm{diag}(1, 1, 1, 1, (x-1), (x-1), (x-1)(x-2), (x-1)(x-2)^2(x-3)).$$
   (a) Give the corresponding decomposition of $\mathbb{C}^8$ regarded as a $\mathbb{C}[x]$-module.
   (b) Give the Jordan Normal form of the matrix A.
   (c) Give the minimal and characteristic polynomials of A.

83. Let $A \in M_8(\mathbb{C})$. Explain how A can be used to define a $\mathbb{C}[X]$-module structure on $\mathbb{C}^8$.

84. Suppose that $XI - A \in M_8(\mathbb{C}[X])$ is equivalent to the matrix
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (X-1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (X-1)(X-2)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (X-1)^2(X-2)^2 \end{pmatrix}.$$
   (i) What is the Jordan normal form of A?
   (ii) What are the minimal and characteristic polynomials of the matrix A?

85. Let V be a complex vector space of dimension 9 and let $T\colon V \to V$ be a linear transformation. Explain how T can be used to make V into a $\mathbb{C}[X]$-module.

86. Let V be a complex vector space of dimension 9 and let $T\colon V \to V$ be a linear transformation. Suppose that, as a $\mathbb{C}[X]$-module,
$$V \cong \frac{\mathbb{C}[X]}{(X-5)^2(X+2)^2} \oplus \frac{\mathbb{C}[X]}{(X+5)^2(X+2)^2}.$$
   (i) What is the Jordan normal form of T?
   (ii) What are the minimal and characteristic polynomials of T?

87. Let V be the $\mathbb{Q}[X]$-module given by $V = \mathbb{Q}[X]^4/N$ where N is the submodule of $\mathbb{Q}[X]^4$ generated by
$$\{(1, 0, 1, 0), (1, X, 0, 0), (1, 0, -X, 0), (-1, 0, 1, x^2)\}.$$
   (i) Find the invariant factor decomposition of V.
   (ii) Write down the primary decomposition of V.

88. Let V be an 8-dimensional complex vector space and let $T\colon V \to V$ be a linear transformation. Explain how V can be regarded as a $\mathbb{C}[X]$-module.

89. Let V be the $\mathbb{C}[X]$-modules given by
$$V = \frac{\mathbb{C}[X]}{(X-2)(X-3)^2} \oplus \frac{\mathbb{C}[X]}{(X-2)(X-3)^3}.$$
Let $T\colon V \to V$ be the linear transformation determined by the action of T.

(i) What is the Jordan Normal Form of T?

(ii) What is the minimal polynomial of T?

(ii) What is the dimension of the eigenspace of T correspondng to the eigenvalue 3?

90. Let $A \in M_{6\times 6}(\mathbb{C})$ such that $xI - A \in M_{6\times 6}(\mathbb{C}[x]$ is equivalent to the diagonal matrix $\mathrm{diag}(1, 1, 1, (x-2), (x-2), (x-2)^2(x-4)^2) \in M_{6\times 6}(\mathbb{C}[x])$.

   (i) What is the Jordan normal form of A?

   (ii) What are the characteristic and minimal polynomials of A?

91. Let V be the $\mathbb{R}[x]$ module given by

$$V = \frac{\mathbb{R}[x]}{(x-1)} \oplus \frac{\mathbb{R}[x]}{(x^2-2)} \oplus \frac{\mathbb{R}[x]}{(x^2+2)}.$$

   (i) Calculate the primary decomposition of V.

   (ii) Calculate the invariant factor decompositions of V.

   (iii) What is the dimension of V when considered as a vector space over $\mathbb{R}$?

92. Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{pmatrix}$.

   (a) Use whichever method you prefer to bring A into Jordan normal form. Carefully record your steps.

   (b) Recall how to use A to equip $\mathbb{C}^3$ with the sturcture of a $\mathbb{C}[x]$-module.

   (c) Write down generators and relations for the $\mathbb{C}[x]$ module encoded by A.

   (d) The structure theorem for module over a PID gives you a different (potentially smaller) set of generators and relations. What is it in this example?

   (e) Find an explicit isomorphism between the representations of parts (c) and (d).

93. Let V be a finite dimensional real vector space and let $T: V \to V$ be a linear transformation. View V as an $\mathbb{R}[X]$-module. Show that V is finitely generated and is a torsion module.

94. Assume that

$$M \cong \frac{\mathbb{R}[X]}{(X^2+1)^2(X-2)} \oplus \frac{\mathbb{R}[X]}{(X^2-1)^2} \oplus \frac{\mathbb{R}[X]}{(X-1)}.$$

   (i) What is the primary decomposition of M?

   (ii) What is the dimension of V as a real vector space?

   (iii) What is the minimal polynomial of T?

95. Let V be a $\mathbb{C}$-vector space with $\dim(V) = 8$ and $T: V \to V$ a linear transformation. Suppose that, as a $\mathbb{C}[t]$-module

$$V \cong \frac{\mathbb{C}[t]}{(t+5)^2\mathbb{C}[t]} \oplus \frac{\mathbb{C}[t]}{(t-3)^3(t+5)^3\mathbb{C}[t]}.$$

   What is the Jordan normal form for the transformation T? What are the eigenvalues of T and how many eigenvectors does T have? What are the minimal and characteristic polynomials of T?

96. Let $R = \mathbb{Q}[X]$ and suppose that the rotsion R-module M is a direct sum of four cyclic modules whose annihilators are

$$(X-1)^3, \qquad (X^2+1)^3, \qquad (X-1)(X^2+1)^4 \qquad \text{and} \qquad (X+2)(X^2+1)^2.$$

Determine the primary decomposition of M and the invariant factor decomposition of M. If M is thought of as a $\mathbb{Q}$-vector space on which X acts as a linear transformation denoted A, determine the mninimal and the characteristic polynomials of A and the dimension of M over $\mathbb{Q}$.

97. Let V be a two dimensional vector space over $\mathbb{Q}$ having basis $\{v_1, v_2\}$. Let T be the linear transformation on V defined by $T(v_1) = 3v_1 - v_2$ and $T(v_2) = 2v_2$. Make V into a $\mathbb{Q}[X]$-module by defining $Xu = T(u)$.

   (a) Show that the subspace $U = \{av_2 \mid a \in \mathbb{Q}\}$ is a $\mathbb{Q}[X]$-submodule of V.

   (b) Let $f = X^2 + 2X - 3 \in \mathbb{Q}[X]$. Determine the vectors fv_1 and fv_2 as linear combinations of v_1 and v_2.

98. Let V be a two dimensional vector space over $\mathbb{Q}$ having basis $\{v_1, v_2\}$. Let T be the linear operator on V defined by $T(v_1) = 3v_1 - v_2$, $T(v_2) = 2v_2$. Recall V (together with T) can be identified with a $\mathbb{Q}[t]$-module by defining $tu = T(u)$.

   (a) Show that the subspace $U = \{av_2 \mid a \in \mathbb{Q}\}$ of V spanned by v_2 is actually a $\mathbb{Q}[t]$-submodule of V.

   (b) Consider the polynomial $f = t^2+2t-3$. Determine the vectors fv_1 and fv_2, that is, express them as linear combinations of v_1 and v_2.

99. Given the matrix $A = \begin{pmatrix} 1-x & 1+x & x \\ x & 1-x & 1 \\ 1+x & 2x & 1 \end{pmatrix} \in M_{3\times 3}(R)$, $R = \mathbb{Q}[x]$, determine the R-module V presented by A. Is V a cyclic R-module? (A module is said to be cyclic if it is generated by a single element).

100. Let $R = \mathbb{Q}[x]$ and suppose that the R-module M is a direct sum of four cyclic modules

$$\frac{\mathbb{Q}[x]}{((x-1)^3)} \oplus \frac{\mathbb{Q}[x]}{((x^2+1)^2)} \oplus \frac{\mathbb{Q}[x]}{((x-1)(x^2+1)^4)} \oplus \frac{\mathbb{Q}[x]}{((x+2)(x^2+1)^2)}.$$

   (a) Decompose M into a direct sum of cyclic modules of the form $\mathbb{Q}[x]/(f_i^{m_i})$, where f_i are monic irreducible polynomials in $\mathbb{Q}[x]$ and $m_i > 0$.

   (b) Find $d_1, d_2, \ldots, d_k \in \mathbb{Q}[x]$ monic polynomials with positive degree such that $d_i | d_{i+1}$, $i = 1, \ldots, k-1$ and $M \cong \mathbb{Q}[x]/(d_1) \oplus \cdots \oplus \mathbb{Q}[x]/(d_k)$.

   (c) Identify the $\mathbb{Q}[x]$-module M with the vector space M over $\mathbb{Q}$ together with a linear operator $X : M \to M, v \mapsto xv$. Suppose the matrix of X is A with respect to a $\mathbb{Q}$-vector space basis of M. Determine the minimal and characteristic polynomials of A and the dimension of M over $\mathbb{Q}$. (the minimal polynomial of A is the smallest degree monic polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(A) = 0$.)

101. Let $V = \mathbb{C}[t]/((t-\lambda)^m)$, $\lambda \in \mathbb{C}$, $m > 0$, be a cyclic $\mathbb{C}[t]$-module.

(a) Show that
$$(w_0 = \bar{1}, w_1 = \overline{t-\lambda}, w_2 = \overline{(t-\lambda)^2}, \ldots, w_{m-1} = \overline{(t-\lambda)^{m-1}})$$
is a basis of V as $\mathbb{C}$-vector space.

(b) Show that the matrix of $T : V \to V, v \mapsto tv$ with respect to the basis in (a) is of the form
$$A = \begin{pmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{pmatrix} \in M_{m\times m}(\mathbb{C}).$$

102. Suppose that V is an 8 dimensional complex vector space and $T : V \to V$ is a linear operator. Using T we make V into a $\mathbb{C}[t]$-module in the usual way. Suppose that as a $\mathbb{C}[t]$-module
$$V \cong \frac{\mathbb{C}[t]}{((t+5)^2)} \oplus \frac{\mathbb{C}[t]}{((t-3)^3(t+5)^3)}.$$
What is the Jordan (normal) form for the transformation T? What are the minimal and characteristic polynomials of T?

103. Let V be an $F[t]$-module and $(v_1, \ldots, v_n)$ a basis of V as an F-vector space. Let $T : V \to V$ be a linear operator and $A \in M_{n\times n}(F)$ the matrix of T with respect to the basis $(v_1, \ldots, v_n)$. Prove that the $F[t]$-matrix $tI - A$ is a presentation matrix of (V, T) regarded as a $F[t]$-module.

104. Determine the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \in M_{3\times 3}(\mathbb{C})$ by decomposing the $\mathbb{C}[t]$-module V presented by the matrix $tI - A \in M_{3\times 3}(\mathbb{C}[t])$.

105. Find all possible Jordan normal forms for a matrix $A \in M_{5\times 5}(\mathbb{C})$ whose characteristic polynomial is $(t+2)^2(t-5)^3$.

106. Let M be the $\mathbb{Q}[x]$-module given by
$$M = \frac{\mathbb{Q}[x]}{(x^2+x+1)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x^3-1)\mathbb{Q}[x]} \oplus \frac{\mathbb{Q}[x]}{(x-3)^2\mathbb{Q}[x]}.$$
Let $T\colon M \to M$ be the $\mathbb{Q}$-linear transformation given by $T(u) = Xu$.

(a) Give the primary decomposition of M as a $\mathbb{Q}[x]$-module.

(b) What is the dimension of M as a vector space over $\mathbb{Q}$?

(c) What is the minimal polynomial of T?

107. Let M be the $\mathbb{C}[x]$-module given by
$$M = \frac{\mathbb{C}[x]}{(x^2+x+1)\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x^3-1)\mathbb{C}[x]} \oplus \frac{\mathbb{C}[x]}{(x-3)^2\mathbb{C}[x]}.$$
Let $T\colon M \to M$ be the $\mathbb{C}$-linear transformation given by $T(u) = Xu$.

(a) Give the primary decomposition of M as a $\mathbb{C}[x]$-module.

(b) What is the Jordan normal form matrix for T?

108. (a) Compute the characteristic polynomial of the following matrix: [as a reminder, the characteristic polynomial of a matrix A is $\det(\lambda I - A)$, which is a polynomial in the variable λ]

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

(b) What is the characteristic polynomial of any matrix in rational canonical form?

(c) Use this to prove the Cayley-Hamilton Theorem: If A is a square matrix and $p(t)$ is its characteristic polynomial, then $p(A) = 0$. [The Cayley-Hamilton theorem holds for matrices with entries in an arbitrary ring, but the intent of this question is to prove it for matrices with entries in a field. However, we can reduce the ring case to the field case (remember how we said to prove $\det(AB) = \det(A)\det(B)$, we could say WLOG R was a field of characteristic zero)]

109. (a) Let V be a vector space over a field k. Let $T : V \to V$ be a linear transformation. Show that by defining $(\sum_i a_i x^i) \cdot v = \sum_i a_i T^i(v)$ defines the structure of a $k[x]$-module on V.

(b) Find an example of a vector space V, together with two linear transformations T and S, such that there does not exist a $k[x, y]$-module structure on V with $x \cdot v = T(v)$ and $y \cdot v = S(v)$ for all $v \in V$.

19.6.3 Smith Normal form

110. Determine the Jordan normal form of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ by calculating the invariant factor matrix of $X - A$.

111. Find all possible Jordan normal forms for a matrices with characteristic polynomial $(t+2)^2(t-5)^3$.

112. Find the Smith normal form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Z}$.

113. Find the rational canonical form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Q}$.

114. Find the Jordan canonical form of $A = \begin{pmatrix} 5 & -4 & 1 \\ -1 & -1 & -2 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{C}$.

115. Find the Smith normal form of $\begin{pmatrix} 11 & -4 & 7 \\ -1 & 2 & 1 \\ 3 & 0 & 3 \end{pmatrix}$ over $\mathbb{Z}$.

116. Let $A = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$. Find $L, R \in GL_3(\mathbb{Z})$ and $d_1, d_2, d_3 \in \mathbb{Z}_{\geq 0}$ such that $d_3\mathbb{Z} \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2, d_3)$.

117. Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$. Find $L, R \in GL_2(\mathbb{Z})$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2)$.

118. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Find $L \in GL_2(\mathbb{Z})$ and $R \in GL_3(\mathbb{Z})$ and $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ such that $d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2)$.

119. Let $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$. Find $L, R \in GL_3(\mathbb{Z})$ and $d_1, d_2, d_3 \in \mathbb{Z}_{\geq 0}$ such that $d_3\mathbb{Z} \subseteq d_2\mathbb{Z} \subseteq d_1\mathbb{Z}$ and $LAR = \mathrm{diag}(d_1, d_2, d_3)$.

120. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} 1-X & 1+X & X \\ X & 1-X & 1 \\ 1+X & 2X & 1 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

121. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} X & 1 & -2 \\ -3 & X+4 & -6 \\ -2 & 2 & X-3 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

122. Let $R = \mathbb{Q}[X]$. Let $A = \begin{pmatrix} X & 0 & 0 \\ 0 & 1-X & 0 \\ 0 & 0 & 1-X^2 \end{pmatrix}$. Find $P, Q \in GL_3(R)$ and $d_1, d_2, d_3 \in \mathbb{Q}[X]_{\mathrm{monic}}$ such that $d_3R \subseteq d_2R \subseteq d_1R$ and $PAQ = \mathrm{diag}(d_1, d_2, d_3)$.

123. Let X be a $n \times m$ matrix with entries in a ring R. Define an ideal $d_1(X)$ to be the ideal in R generated by all entries of X. Let A and B be invertible matrices (of the appropriate sizes) with entries in R. Prove that $d_1(AXB) = d_1(X)$.

124. With notation as in Question 123, let $d_k(X)$ be the ideal in R generated by all $k \times k$ minors in X. Prove that $d_k(AXB) = d_k(X)$.

125. Use the previous result to show that the elements d_i in Smith Normal Form are unique up to associates.