19.5.1 Function Fields

1. Let $K = \mathbb{C}(t)$. Let $E = \mathbb{C}(t^2)$ and $F = \mathbb{C}(t^2 - t)$.

   (a) Find field automorphisms σ and τ of K such that σ fixes E, τ fixes F and such that $\sigma\tau$ is of infinite order.

   (b) Prove that $E \cap F = \mathbb{C}$.

2. Let $K = \mathbb{C}(t)$. Let n be a positive integer and let $u = t^n + t^{-n}$. Define automorphisms σ and τ of K by $\sigma(t) = \zeta t$ and $\tau(t) = t^{-1}$, where $\zeta = e^{\frac{2\pi i}{n}}$.

   (a) Prove that $\mathbb{C}(u)$ is fixed by both σ and τ.

   (b) Find the minimal polynomial for t over the field $\mathbb{C}(u)$.

   (c) Prove that K is a Galois extension of $\mathbb{C}(u)$.

3. (a) Let $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Prove that there exists an automorphism σ of $\mathbb{C}(z)$ with $\sigma(z) = \frac{az+b}{cz+d}$ (these are called Mobius transformations)

   (b) Determine the relationship between composition of Mobius transformations and matrix multiplication.

   (c) Show that the automorphisms $\sigma(t) = it$ and $\tau(t) = t^{-1}$ of $\mathbb{C}(t)$ generate a group G that is isomorphic to the dihedral group D_4.

   (d) Let $u = t^4 + t^{-4}$. Show that u is fixed under H.

   (e) What is $[\mathbb{C}(t) : \mathbb{C}(u)]$?

4. Let $F = \mathbb{C}(w)$. Let $f(x) = x^4 - 4x^2 + 2 - w$.

   (a) Prove that $f(x)$ is irreducible in $F[x]$. [Hint: Gauss' Lemma]

   (b) Let $K = F[x]/(f(x))$. Prove that K is not a splitting field of f. [Hint: It may be easier to identify $w = t^4 + t^{-4}$ and identify F with the corresponding subfield of $\mathbb{C}(t)$, as here you can compute the roots of f explicitly]

5. Let $K = \mathbb{C}(t)$. Define automorphisms σ and τ of K by $\sigma(t) = 1 - t$ and $\tau(t) = \frac{1}{t}$. Let

$$w = \frac{(t^2 - t + 1)^3}{t^2(t-1)^2} \qquad \text{and} \qquad F = \mathbb{C}(w).$$

   (a) Prove that $\sigma(w) = w$ and $\tau(w) = w$.

   (b) Find a polynomial $f \in F[x]$ of degree 6 which has t as a root. What are the other 5 roots of f in K?

   (c) Let G be the group generated by the automorphisms σ and τ. Prove that $F = K^G$. You may use without proof that $G \cong S_3$, the symmetric group on 3 letters.

   (d) How many fields are there with $F \subseteq E \subseteq K$?

   (e) How many of the fields from part (d) are Galois extensions of F?