19.6.4 Free, torsion-free, annihilators, torsion

126. Let M be an R-module and let $m \in M$. Show that $\mathrm{ann}(m)$ is an ideal in R.

127. Let I be an ideal in R. Show that $\mathrm{ann}(R/I) = I$.

128. Let M_1 and M_2 be R-modules. Show that $\mathrm{ann}(M_1 \oplus M_2) = \mathrm{ann}(M_1) \cap \mathrm{ann}(M_2)$.

129. Give the definition of the torsion submodule of an R-module.

130. Give the definition of the torsion submodule of an R-module.

131. Let M be an R-module. Show that $\mathrm{Tor}(M)$ is a submodule of M.

132. Define what it means to say that a module is torsion free.

133. Let R be a commutative ring with identity. What does it mean to say that an R-module is free?

134. What does it mean to say that an R-module is free?

135. Let R be a commutative ring with identity. What does it mean to say that an R-module is free?

136. Let M be a module. Define carefully what it means to say that M is free.

137. What does it mean to say that an R-module is free?

138. Let M be an R-module. Give the definitions of what it means to say that M is torsion free and what it means to say that M is free.

139. Show that $R\text{-span}(S) = \{r_1v_1 + \cdots r_kv_k \mid k \in \mathbb{Z}_{>0},\, r_1, \ldots, r_k \in R \text{ and } v_1, \ldots, v_k \in S\ \}$.

140. Let M be an R-module. Prove that a subset S of M is a basis of M if and only if every element of M can be written uniquely as a linear combination of elements from S.

141. Let F and G be two free R-modules of rank m and n respectively. Show that the R-module $F \oplus G$ is free of rank $m + n$.

142. Let R be a ring and let V be a free module of finite rank over R.
    (a) Show that every set of generators of V contains a basis of V.
    (b) Show that every linearly independent set in V can be extended to a basis of V.

143. Show that every finitely generated R-module is isomorphic to a quotient of a free R-module.

144. Show that every finitely generated R-module is isomorphic to a quotient of a free R-module.

145. (a) Give the definitions of a module and a free module.
    (b) Give an example of a free module having a proper submodule of the same rank.
    (c) Show that, as a $\mathbb{Z}$-module, $\mathbb{Q}$ is torsion free but not free.

146. Show that $\mathbb{Q}$, considered as a $\mathbb{Z}$-module, is not free.

147. Show that $\mathbb{Q}$ considered as a $\mathbb{Z}$-module, is torsion free but not free.

148. Let $R = \mathbb{R}[X, Y]$ and let $I = (X, Y)$ be the ideal generated by X and Y. Show that I considered as an R-module is not free.

149. Let $R = \mathbb{R}[X, Y]$ and let $I = (X, Y)$ be the ideal generated by X and Y. Show that I considered as an R-module is not free.

150. Let $R = \mathbb{Z}/6\mathbb{Z}$ and let $F = R^{\oplus 2}$. Write down a basis of F. Let $N = \{(0, 0), (3, 0)\}$. Show that N is a submodule of the free module F and N is not free.

151. Give an example of a submodule of a free module that is not free.

152. Give an example of a finitely generated R-module that is torsion-free but not free.

153. Give an example of a free module M and a generating set $S \subseteq M$ such that M does not contain a basis.

154. Let R be a commutative unital ring, let F be a free R-module and let $\varphi\colon M \to F$ be a surjective module homomorphism. Show that $M \cong F \oplus \ker(\varphi)$.

155. Suppose that R is an integral domain and M is an R-module. Let T be the torsion submodule of M. Show that the R-module M/T is torsion free.

156. Suppose that R is an integral domain and M is an R-module. Let T be the torsion submodule of M. Show that the R-module M/T is torsion free.

157. Let R be an integral domain. Show that a free R-module is torsion free.

158. Show that if R is an integral domain and M is free then M is torsion free.

159. Let R be a integral domain and let M be a free R-module. Show that M is torsion free.

160. Give an example of an integral domain R and an R-module M such that M is torsion free and M is not free.

161. Show that R is a torsion free R-module if and only if R is an integral domain.

162. Show that $\mathbb{Q}$ as a $\mathbb{Z}$-module is torsion free but not free.

163. Let R be an integral domain. Let I be an ideal in R. Show that I is a free R-module if and only if it is principal.

164. Let R be an integral domain. Let V be a free R-module of rank d. Define $\mathrm{End}_R(V)$, explain (with proof) how it is a ring, and show that $\mathrm{End}_R(V) \cong M_{d\times d}(R)$.

165. Let R be an integral domain. Let V be a free R-module with basis $\{v_1, \ldots, v_d\}$. Let $\varphi\colon V \to V$ be an R-module morphism. Prove that $\{\varphi(v_1), \ldots, \varphi(v_d)\}$ is a basis of V if and only if φ is an isomorphism.

166. State the structure theorem for finitely generated modules over a principal ideal domain.

167. State the structure theorem for finitely generated modules over a PID.

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172. State the structure theorem for finitely generated modules over a PID.

173. State carefully the invariant factor theorem which describes the structure of finitely generated modules over a principal ideal domain.

174. Describe the primary decomposition of a finitely generated torsion module over a PID.

175. Let M be a finitely generated torsion module over a PID R. Show that M is indecomposable if and only if $M = Rx$ where $\mathrm{ann}_R(z) = (p^e)$ and p is a prime of R.

176. Use the structure theorem for modules to show that a torsion free finitely generated module over a PID is free.

177. Show that if R is a PID then any finitely generated and torsion free R module is free.