3.14 Lecture 17: Finiteness conditions and the Jordan-Hölder theorem

Let R be a ring and let M be an R-module.

- The **lattice of submodules of** M is

$$\mathcal{S}_0^M = \{\text{submodules of } M\} \qquad \text{partially ordered by inclusion.}$$

- The R-module M **satisfies ACC** if increasing sequences in $\mathcal{S}_0^M$ are finite.
- The R-module M **satisfies DCC** if decreasing sequences in $\mathcal{S}_0^M$ are finite.
- The R-module is **simple** if the only submodules of M are 0 and M.
- A **finite composition series of** M is a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \qquad \text{such that } M_i/M_{i+1} \text{ is simple and } n \in \mathbb{Z}_{>0}.$$

- The R-module M is **finitely generated** if there exists $k \in \mathbb{Z}_{>0}$ and $m_1, \ldots, m_k \in M$ such that

$$M = Rm_1 + \cdots + Rm_k.$$

Proposition 3.65. *Let N be a submodule of M.*

(a) M satisfies ACC if and only if N and M/N satisfy ACC.

(b) M satisfies DCC if and only if N and M/N satisfy DCC.

(c) M satsfies both ACC and DCC if and only if N and M/N satisfy both ACC and DCC.

Proposition 3.66. *Let R be a ring and let M be an R-module.*

(a) If M is finitely generated and N is an R-submodule of M then M/N is finitely generated.

(b) M satisfies ACC if and only if every submodule of M is finitely generated.

(c) If R satisfies ACC and M is finitely generated then M satisfies ACC.

(e) If R satisifes DCC and M is finitely generated then M satisfies both ACC and DCC.

Theorem 3.67. *(Jordan-Hölder theorem) Let A be a ring and let M be an A-module.*

(a) M has a finite composition series if and only if M satisfies ACC and DCC.

(b) Any two series

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \qquad \text{and} \qquad 0 \subseteq M_1' \subseteq M_2' \subseteq \cdots \subseteq M_s' = M$$

can be refined to have the same length and the same composition factors.

Greedy refinement:. Assume that

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{r-1} \overset{p}{\subseteq} M_r = M \qquad \text{and} \qquad 0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_{s-1} \overset{q}{\subseteq} N_s = M$$

are composition series of M. Then build the series

$$0 \subseteq M_1 \cap N_{s-1} \subseteq M_2 \cap N_{s-1} \subseteq \cdots \subseteq M_{r-1} \cap N_{s-1} \overset{p}{\subseteq} N_{s-1} \overset{q}{\subseteq} M_r = M.$$

This takes the q factor out of the series of (M_i) and moves it to the end.

Symmetric refinement: Let

$$0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \qquad \text{and} \qquad 0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_s = M$$

be finite ascending chains in $\mathcal{S}_{[0,M]}$. For $I \in \{1, \dots, r\}$ and $j \in \{1, \dots, s\}$ define

$$M_{ij} = (M_i + N_j) \cap M_{i+1} \qquad \text{and} \qquad N_{ji} = N_j + M_i) \cap N_{j+1}.$$

This expands $M_i \subseteq M_{i+1}$ to

$$M_i = (N_0' + M_i) \cap M_{i+1} \subseteq (N_1' + M_i) \cap M_{i+1} \subseteq \cdots \subseteq (N_s' + M_i) \cap M_{i+1} = M_{i+1},$$

and $N_j \subseteq N_{j+1}$ to

$$N_j = (M_0 + N_j) \cap N_{j+1} \subseteq (M_1 + N_j) \cap N_{j+1} \subseteq \cdots \subseteq (M_r + N_j) \cap N_{j+1} = N_{j+1}.$$

Let

$$Q_{ij} = \frac{M_{ij}}{M_{i,j-1}} \qquad \text{and} \qquad Q'_{ji} = \frac{N_{ji}}{N_{j,i-1}}.$$

Then

$$Q_{ij} \cong Q'_{ji}$$

and so the two new chains (M_{ij}) and (N_{ji}) have the same length and the same multiset of factors.

Example:. Two factorizations of $d = 2^2 3^3$ in $\mathbb{Z}$ are

$$\big(2^2 3^3 \mathbb{Z} \subseteq 2^2 3^2 \mathbb{Z} \subseteq 2^2 \mathbb{Z} \subseteq \mathbb{Z}\big) = (M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3)$$

and

$$\big(2^2 3^3 \mathbb{Z} \subseteq 3^3 \mathbb{Z} \subseteq \mathbb{Z}\big) = (N_0 \subseteq N_1 \subseteq N_2).$$

Then

$$\begin{pmatrix} 2^2 3^3 \mathbb{Z} & \overset{1}{\subseteq} & 2^2 3^3 \mathbb{Z} & \overset{3}{\subseteq} & 2^2 3^2 \mathbb{Z} \\ 2^2 3^2 \mathbb{Z} & \overset{1}{\subseteq} & 2^2 3^2 \mathbb{Z} & \overset{3^2}{\subseteq} & 2^2 \mathbb{Z} \\ 2^2 \mathbb{Z} & \overset{2^2}{\subseteq} & \mathbb{Z} & \overset{1}{\subseteq} & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} M_{00} & \subseteq & M_{01} & \subseteq & M_{02} \\ M_{10} & \subseteq & M_{11} & \subseteq & M_{12} \\ M_{20} & \subseteq & M_{21} & \subseteq & M_{22} \end{pmatrix}$$

and

$$\begin{pmatrix} 2^2 3^3 \mathbb{Z} & \overset{1}{\subseteq} & 2^2 3^3 \mathbb{Z} & \overset{1}{\subseteq} & 2^2 3^3 \mathbb{Z} & \overset{2^2}{\subseteq} & 3^3 \mathbb{Z} \\ 3^3 \mathbb{Z} & \overset{3}{\subseteq} & 3^2 \mathbb{Z} & \overset{3^2}{\subseteq} & \mathbb{Z} & \overset{1}{\subseteq} & \mathbb{Z} \end{pmatrix} = \begin{pmatrix} N_{00} & \subseteq & N_{01} & \subseteq & N_{02} & \subseteq & N_{03} \\ N_{10} & \subseteq & N_{11} & \subseteq & N_{12} & \subseteq & N_{13} \end{pmatrix}$$

and the succesive quotients of these two series are related by

$$\begin{pmatrix} 1 & 3 \\ 1 & 3^2 \\ 2^2 & 1 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 2^2 \\ 3 & 3^2 & 1 \end{pmatrix}.$$

3.14.1 Some proofs

Proposition 3.68. *Let N be a submodule of M.*

(a) M satisfies ACC if and only if N and M/N satisfy ACC.

(b) M satisfies DCC if and only if N and M/N satisfy DCC.

(c) M satsfies both ACC and DCC if and only if N and M/N satisfy both ACC and DCC.

Proof. (a) $\Rightarrow$: Assume that M satisfies ACC.
To show: (aa) N satisfies ACC.
To show: (ab) M/N satsfies ACC.

(aa) Let $0 = N_0 \subseteq N_1 \subseteq \cdots$ be a chain in $\mathcal{S}_N$.
Since $N \subseteq M$ then $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq M$ is a chain in $\mathcal{S}_M$.
Since M satisfies ACC then $0 = N_0 \subseteq N_1 \subseteq \cdots$ is finite.
So N satisfies ACC.

(ab) Let $0 = M_0/N \subseteq M_1/N \subseteq \cdots \subseteq M/N$ be a chain in $\mathcal{S}_{M/N}$.
By the correspondence theorem the chain in $\mathcal{S}_{M/N}$ corresponds to a chain $0 \subseteq N = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ in $\mathcal{S}_M$.
Since M satsifes ACC then $0 \subseteq N = M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ is finite.
So $0 = M_0/N \subseteq M_1/N \subseteq \cdots \subseteq M/N$ is finite.
So M/N satsfies ACC.

(a) $\Leftarrow$: Assume that N and M/N satisfy ACC.
To show: M satsifies ACC. Let $0 = M_0 \subseteq M_1 \subseteq \cdots$ be an ascending chain in $\mathcal{S}_0^M$.
Then

$$0 = \frac{M_0 + N}{N} \subseteq \frac{M_1 + N}{N} \subseteq \cdots \subseteq \frac{M}{N} \qquad \text{and} \qquad 0 = (M_0 \cap N) \subseteq (M_1 \cap N) \subseteq \cdots \subseteq N$$

are ascending chains in $\mathcal{S}_0^{M/N}$ and $\mathcal{S}_0^N$.
Let $k \in \mathbb{Z}_{>0}$ such that if $\ell \in \mathbb{Z}_{\geq k}$ then

$$\frac{M_\ell + N}{N} = \frac{M_k + N}{N} \qquad \text{and} \qquad M_\ell \cap N = M_k \cap N.$$

By the correspondence theorem, if $\ell \in \mathbb{Z}_{\geq k}$ then

$$M_\ell + N = M_k + N \qquad \text{and} \qquad M_\ell \cap N = M_k \cap N.$$

Thus

$$M_\ell \cap (M_k + N) = M_\ell \cap (M_\ell + N) = M_\ell \qquad \text{and} \qquad M_k + (m_\ell \cap N) = M_k + (M_k \cap N) = M_k.$$

Since $M_k \subseteq M_\ell$ then the modular law says that

$$M_\ell \cap (M_k + N) = M_k + (M_\ell \cap N).$$

So $M_k = M_\ell$.
(b) The proof of (b) is similar to the proof of (a),
except with ACC replaced by DCC and $\subseteq$ replaced by $\supseteq$.
(c) is the combination of (a) and (b). □

Proposition 3.69. *Let R be a ring and let M be an R-module.*

(a) If M is finitely generated and N is an R-submodule of M then M/N is finitely generated.

(b) M satisfies ACC if and only if every submodule of M is finitely generated.

(c) If R satisfies ACC and M is finitely generated then M satisfies ACC.

(e) If R satisifes DCC and M is finitely generated then M satisfies both ACC and DCC.

Proof. (a) If $m_1, \ldots, m_k$ are generators of M then $m_1 + N, \ldots, m_k + N$ are generators of M/N.

(b) $\Leftarrow$: Assume that every submodule of M is finitely generated.
Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M.
To show: There exists $r \in \mathbb{Z}_{>0}$ such that if $\ell \in \mathbb{Z}_{\geq r}$ then $N_\ell = N_r$.
Then $N_{\mathrm{un}} = \bigcup_{i \in \mathbb{Z}_{>0}} N_i$ is a finitely generated submodule of M
Let $x_1, \ldots, x_k$ be generators of N_{un} and let $\ell_1, \ldots, \ell_k$ be such that $x_i \in N_{\ell_i}$.
Then $x_1, \ldots, x_k \in N_r$ where $r = \max\{\ell_1, \ldots, \ell_k\}$.
So $N_{\mathrm{un}} = \bigcup_{i \in \mathbb{Z}_{>0}} N_i = N_r$ and if $\ell \in \mathbb{Z}_{>r}$ then $N_r = N_\ell$.
So M satisfies ACC.

(b) $\Rightarrow$: Assume that M satisfies ACC and let N be a submodule of M. Then one of the equivalent characterizations of ACC gives that the set of finitely generated submodule of N,

$$\{P \subseteq N \mid P \text{ is finitely generated}\}, \qquad \text{has a maximal element } P_{\max}.$$

To show: $N = P_{\max}$.
By definition, $P_{\max} \subseteq N$.
To show: $N \subseteq P_{\max}$.

Let $x \in N$.

Then $P_{\max} + \mathbb{A}x \subseteq N$ and $P_{\max} + \mathbb{A}x$ is finitely generated.

So $P_{\max} + \mathbb{A}x \subseteq P_{\max}$.

So $x \in P_{\max}$.

So $N \subseteq P_{\max}$.

So $N = P_{\max}$.
So N is finitely generated.

(c) Assume R satisfies ACC and M is finitely generated.
To show: M satsfies ACC.
Since M is finitely generated there exists $n \in \mathbb{Z}_{>0}$ and a surjective homomorphism $\mathbb{A}^{\oplus v} \to M$.
Since $\mathbb{A}$ satisfies ACC then $\mathbb{A}^{\oplus n}$ satisifes ACC.
So there is an exact sequence $0 \to K \to \mathbb{A}^{\oplus n} \to M \to 0$ with $\mathbb{A}^{\oplus n}$ satisfiying ACC.
By Proposition 4.4(a), K and M satisfy ACC.
So M satisfies ACC.

(da) To show: If R satisfies DCC and M is finitely generated then M satisfies DCC. The proof of (da) is the same as the proof of (c) except with ACC replaced by DCC and the inclreasing chains replaced by decreasing chains.

(db) Assume R satisfies DCC and M is finitely generated. To show: M satisfies ACC.
Let $M_i = \mathrm{Rad}(R)^i M$.
By (da), M satisfies DCC, and so M_i and M/M_i satisfy DCC.
So M_i/M_{i+1} satisfies DCC and $\mathrm{Rad}(R)$ acts on M_i/M_{i+1} by 0.
So M_i/M_{i+1} is a $R/\mathrm{Rad}(R)$-module and thus M_i/M_{i+1} is a finite direct sum of simple submodules.
So, by (a), M has a composition series and satisfies both ACC and DCC. $\square$