6.2.3 Fields of fractions

Definition. Let R be an integral domain.

- A **fraction** is an expression $\dfrac{a}{b}$ with $a \in R$, $b \in R$ and $b \neq 0$.

Proposition 6.5. *Let R be an integral domain. Let $F_R = \left\{\dfrac{a}{b} \mid a, b \in R, b \neq 0\right\}$ be the set of fractions. Define two fractions $\frac{a}{b}$, $\frac{c}{d}$ to be equal if $ad = bc$, i.e.*

$$\frac{a}{b} = \frac{c}{d} \quad \text{if } ad = bc.$$

Then equality of fractions is an equivalence relation on F_R.

Proposition 6.6. *Let R be an integral domain. Let $F_R = \left\{\dfrac{a}{b} \mid a, b \in R, b \neq 0\right\}$ be its set of fractions with equality of fractions be as defined in Proposition 6.5. Then the operations $+\colon F_R \times F_R \to F$ and $\times\colon F_R \times F_R \to F_R$ given by*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{are well defined.}$$

Theorem 6.7. *Let R be an integral domain and let $F_R = \left\{\dfrac{a}{b} \mid a \in R, b \in R - \{0\}\right\}$ be the set of fractions with equality of fractions be as defined in Proposition 6.5 and let operations $+\colon F_R \times F_R \to F_R$ and $\times\colon F_R \times F_R \to F_R$ be as given in Proposition 6.6. Then F_R is a field.*

Definition. Let R be an integral domain.

- The **field of fractions** of R is the set $F_R = \{\dfrac{m}{n} \mid m, n \in R, n \neq 0\}$ of fractions with **equality of fractions** defined by
$$\frac{m}{n} = \frac{p}{q} \quad \text{if } mq = np$$
and operations of **addition** $+\colon F_R \times F_R \to F_R$ and **multiplication** $\times\colon F_R \times F_R \to F_R$ defined by
$$\frac{m}{n} + \frac{p}{q} = \frac{mq+np}{pq} \qquad \text{and} \qquad \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq}.$$

HW: Give an example of an integral domain R and its field of fractions.

Proposition 6.8. *Let R be an integral domain with identity 1 and let F_R be its field of fractions. Then the map $\varphi\colon R \to F_R$ given by*

$$\begin{array}{rccc} \varphi\colon & R & \to & F_R \\ & r & \mapsto & \frac{r}{1} \end{array}$$

is an injective ring homomorphism.