6.2 Fields, Integral Domains, Fields of Fractions

6.2.1 R/M is a field $\iff$ M is a maximal ideal.

Definition.

- A **field** is a commutative ring F such that if $x \in F$ and $x \neq 0$ then there exists an element $x^{-1} \in F$ such that $xx^{-1} = 1$.
- A **proper ideal** is an ideal of R that is not the zero ideal (0) and not the whole ring R.
- A **maximal ideal** is an ideal M of a ring R such that
  (a) $M \neq R$,
  (b) If M' is an ideal of R and $M \subseteq M' \neq R$ then $M = M'$.

Lemma 6.1. *Let F be a commutative ring. Then F is a field if and only if the only ideals of F are $\{0\}$ and F.*

Theorem 6.2. *Let R be a commutative ring and let M be an ideal of R. Then*

$$R/M \text{ is a field} \quad \text{if and only if} \quad M \text{ is a maximal ideal.}$$

6.2.2 R/P is an integral domain $\iff$ P is a prime ideal.

Definition.

- An **integral domain** is a commutative ring R such that if $a, b \in R$ and $ab = 0$ then either $a = 0$ or $b = 0$.
- A **zero divisor** in a ring R is an element $a \in R$ such that there exists $b \in R$ with $\neq 0$ and $ab = 0$.
- A **prime ideal** is an ideal P in a commutative ring R such that if $a, b \in R$ and $ab \in P$ then either $a \in P$ or $b \in P$.

HW: Show that an integral domain is a commutative ring with no zero divisors except 0.

Proposition 6.3. (Cancellation Law) *Let R be an integral domain. If $a, b, c \in R$ and $c \neq 0$ and $ac = bc$ then $a = b$.*

Theorem 6.4. *Let R be a commutative ring and let P be an ideal of R. Then*

$$R/P \text{ is an integral domain} \quad \text{if and only if} \quad P \text{ is a prime ideal.}$$