6.4 Euclidean Domains, PIDs and UFDs

6.4.1 R is a Euclidean domain $\Longrightarrow$ R is a PID

Definition. Let $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers.

- A **Euclidean domain** is an integral domain R with a function

  $$\sigma\colon R - \{0\} \to \mathbb{Z}_{\geq 0}, \qquad \text{a } \textbf{size function}$$

  such that if $a, b \in R$ and $a \neq 0$ then there exist $q, r \in R$ such that

  $$b = aq + r, \qquad \text{where either } r = 0 \text{ or } \sigma(r) < \sigma(a).$$

- Let R be a commutative ring. A **principal ideal** is an ideal generated by a single element.

- A **principal ideal domain** (or **PID**) is an integral domain for which every ideal is principal.

Theorem 6.11. *If R is a Euclidean domain then R is a principal ideal domain.*

HW: Show that $\mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{-19}]$ is a PID that is not a Euclidean domain.

6.4.2 R is a PID $\Longrightarrow$ R is a UFD

Definition. Let R be an integral domain.

- A **unit** is an element $a \in R$ such that $aR = R$.
- An element $p \in R$ is **irreducible** if pR if $p \neq 0$, $pR \neq R$ and R/pR is a simple R-module.
- A **unique factorization domain** (or **UFD**) is an integral domain R such that
  (a) If $x \in R$ then there exist irreducible $p_1, \ldots, p_n \in R$ such that $x = p_1 \cdots p_n$.
  (b) If $x \in R$ and $x = p_1 \cdots p_n = uq_1 \cdots q_m$ where $u \in R$ is a unit and $p_1, \ldots, p_n,\ q_1, \ldots, q_m \in R$ are irreducible then $m = n$ and there exists a permutation $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and units $u_1, \ldots, u_n \in R$ such that

  $$\text{if } i \in \{1, \ldots, n\} \text{ then } q_i = u_i p_{\sigma(i)}.$$

The following theorem is a consequence of the Jordan-Hölder Theorem.

Theorem 6.12. *If R is a principal ideal domain then R is a unique factorization domain.*

HW: Show that $\mathbb{C}[x, y]$ and $\mathbb{Z}[x]$ are UFDs that are not PIDs.

HW: Show that if R is a PID and $p \in R$ then p is irreducible if and only if pR is a maximal ideal.

HW: Show that if R is a UFD and $p \in R$ is irreducible then pR is a prime ideal.