Linear Algebra
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Span

Let V be an $\mathbb{R}$-vector space. Let $k \in \mathbb{Z}_{>0}$.
Let $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of V.

$$\mathrm{span}(S) = \{c_1 v_1 + \cdots + c_k v_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

Topic 4 Example 16 Let

$$S = \{|1,-1\rangle, |2,4\rangle\} \text{ in } \mathbb{R}^2.$$

Show that $\mathrm{span}(S) = \mathbb{R}^2$.

To show: (a) $\mathrm{span}(S) \subseteq \mathbb{R}^2$

(b) $\mathbb{R}^2 \subseteq \mathrm{span}(S)$

(a) Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar multiplication then $\mathrm{span}(S) \subseteq \mathbb{R}^2$.

(b) To show: $\mathbb{R}^2 \subseteq \mathrm{span}(S)$.

To show: $\mathrm{span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathrm{span}(S)$

Since $\mathrm{span}(S)$ is closed under addition and scalar multiplication

To show $\{|1,0\rangle, |0,1\rangle\} \subseteq \mathrm{span}(S)$

To show: There exist $c_1, c_2 \in \mathbb{R}$ and $d_1, d_2 \in \mathbb{R}$ such that

$$c_1|1,-1\rangle + c_2|1,4\rangle = |1,0\rangle$$

$$d_1|1,-1\rangle + d_2|2,4\rangle = |0,1\rangle.$$

To show: there exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$\frac{2}{3}|1,-1\rangle + \frac{1}{6}|2,4\rangle = |1,0\rangle$$

$$\frac{-1}{3}|1,-1\rangle + \frac{1}{6}|2,4\rangle = |0,1\rangle.$$

So $|1,0\rangle \in \mathrm{span}(S)$ and $|0,1\rangle \in \mathrm{span}(S)$

So $\mathrm{span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathrm{span}(S)$.

So $\mathbb{R}^2 \subseteq \mathrm{span}(S)$.

So $\mathrm{span}(S) = \mathbb{R}^2$.

<u>Topic 4 Example 18</u> Let

$$S = \{1+x+x^2, x^2\} \text{ in } \mathbb{P}_2,$$

where $\mathbb{P}_2 = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$.

Show that $\mathrm{span}(S) = \mathbb{P}_2$.

To show: (a) $\mathrm{span}(S) \subseteq \mathbb{P}_2$

(b) $\mathbb{P}_2 \subseteq \mathrm{span}(S)$

(a) Since $S \subseteq \mathbb{P}_2$ and $\mathbb{P}_2$ is closed under addition and scalar multiplication then $\mathrm{span}(S) \subseteq \mathbb{P}_2$.

(b) To show: $\mathbb{P}_2 \subseteq \mathrm{span}(S)$.

To show: $\mathrm{span}\{1, x, x^2\} \subseteq \mathrm{span}(S)$.

Since $\mathrm{span}(S)$ is closed under addition and scalar multiplication then

To show: $\{1, x, x^2\} \subseteq \mathrm{span}(S)$.

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$c_1(1+x+x^2)+c_2x^2=1,$$

$$d_1(1+x+x^2)+d_2x^2=x,$$

$$r_1(1+x+x^2)+r_2x^2=x^2.$$

To show: there exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply both sides by $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since the top row on the left side is all 0 and the top row on the right side is not all 0 then there do not exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

So $\{1, x, x^2\} \not\subseteq \mathrm{span}(S)$.

So $\operatorname{span}(S) \neq P_2$.