n-dimensional space

Let $n \in \mathbb{Z}_{>0}$ and let $\mathbb{R}^n = M_{n\times 1}(\mathbb{C})$.

Let $$|a_1, \dots, a_n\rangle = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$$

Dirac used the notation
$$\langle a_1, \dots, a_n| = |a_1, \dots, a_n\rangle^t = (a_1\ a_2\ \cdots\ a_n) \in M_{1\times n}(\mathbb{R})$$

Let $a = |a_1, \dots, a_n\rangle$ and $b = |b_1, \dots, b_n\rangle$ in $\mathbb{R}^n$.
$$b - a = |b_1 - a_1, \dots, b_n - a_n\rangle.$$

The <u>standard inner product of a and b</u> is
$$\langle a|b\rangle = \langle a_1, \dots, a_n | b_1, \dots, b_n\rangle = (a_1\ \cdots\ a_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

So
$$\langle a_1, \dots, a_n | b_1, \dots, b_n\rangle = a_1 b_1 + \cdots + a_n b_n$$

The <u>length of a</u> is
$$\|a\| = \sqrt{\langle a|a\rangle} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

The <u>distance between a and b</u> is
$$d(a,b) = \|b-a\| = \sqrt{(b_1-a_1)^2 + \cdots + (b_n-a_n)^2}$$

The <u>unit vector in the direction of a</u> is
$$\hat{a} = \frac{1}{\|a\|}\, a$$

The projection of b onto the direction of a is

$$\operatorname{proj}_a(b) = \frac{1}{\|a\|}\langle a|b\rangle \frac{a}{\|a\|}$$

The angle between a and b is $\theta \in \mathbb{R}_{[0,\pi]}$ determined by $\cos(\theta) = \frac{1}{\|a\|\cdot\|b\|}\langle a|b\rangle$.

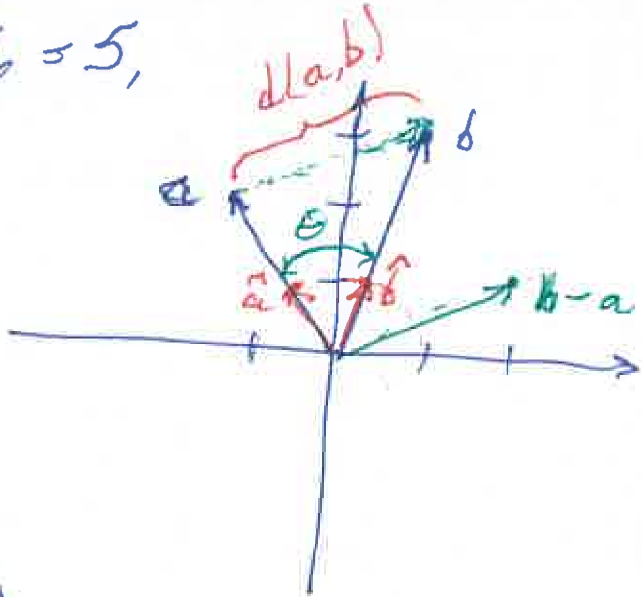
2 dimensional space example

$$a = \begin{pmatrix}-1\\2\end{pmatrix}, \quad b = \begin{pmatrix}1\\3\end{pmatrix}, \quad b-a = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\langle a|b\rangle = (-1)\cdot 1 + 2\cdot 3 = -1+6 = 5,$$

$$\|a\| = \sqrt{(-1)^2+2^2} = \sqrt{5},$$

$$\|b\| = \sqrt{1^2+3^2} = \sqrt{10},$$



$$\hat{a} = \frac{1}{\sqrt{5}}a = \begin{pmatrix}-1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix} = \left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$$

$$\hat{b} = \frac{1}{\sqrt{10}}b = \begin{pmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{pmatrix} = \left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\rangle$$

$$d(a,b) = \|b-a\| = \sqrt{2^2+1^2} = \sqrt{5}$$

$$\operatorname{proj}_a(b) = \frac{1}{\sqrt{5}}\langle a|b\rangle\frac{1}{\sqrt{5}}a = \frac{1}{5}\cdot 5a = \langle -1, 2\rangle$$

Since $\cos(\theta) = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}}\cdot 5 = \frac{1}{\sqrt{2}}$ then $\theta = \frac{\pi}{4}$

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3 dimensional space example

$a = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \langle 2, -1, -2 \rangle$, $\quad b = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \langle 2, 1, 3 \rangle$,

$$b - a = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} = \langle 0, 2, 5 \rangle.$$

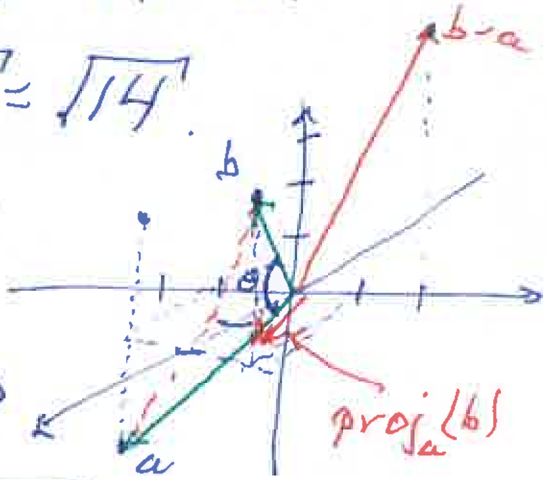
$\langle a | b \rangle = 2 \cdot 2 + (-1) \cdot 1 + (-2) \cdot 3 = 4 - 1 - 6 = -3.$

$\|a\| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3.$

$\|b\| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{4+1+9} = \sqrt{14}.$

$\hat{a} = \frac{1}{3} a = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle$

$\hat{b} = \frac{1}{\sqrt{14}} b = \left\langle \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle$



$d(a,b) = \|b - a\| = \sqrt{0^2 + 2^2 + 5^2} = \sqrt{29}$

$\mathrm{proj}_a(b) = \frac{1}{3} \langle a | b \rangle \cdot \frac{1}{3} a = \frac{1}{9}(-3) \langle 2, -1, -2 \rangle$

$= \left\langle -\frac{2}{3}, \frac{1}{3}, +\frac{2}{3} \right\rangle$

Since $\cos(\theta) = \frac{1}{3} \cdot \frac{1}{\sqrt{14}} \langle a | b \rangle = \frac{1}{3}(-3)\frac{1}{\sqrt{14}} = \frac{-1}{\sqrt{14}}$

then $\quad \theta = 1.84134608977\ldots$

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4-dimensional space example

$a = \langle 0, 2, 2, -1 \rangle$ and $b = \langle -1, 1, 1, -1 \rangle$.

Then $b - a = \langle -1, -1, -1, 0 \rangle$

$\|a\| = \sqrt{0^2 + 2^2 + 2^2 + (-1)^2} = \sqrt{0+4+4+1} = \sqrt{9} = 3$

$\|b\| = \sqrt{(-1)^2 + 1^2 + 1^2 + (-1)^2} = \sqrt{4} = 2$

$d(a,b) = \sqrt{(-1)^2 + (-1)^2 + (-1)^2 + 0^2} = \sqrt{3}$.

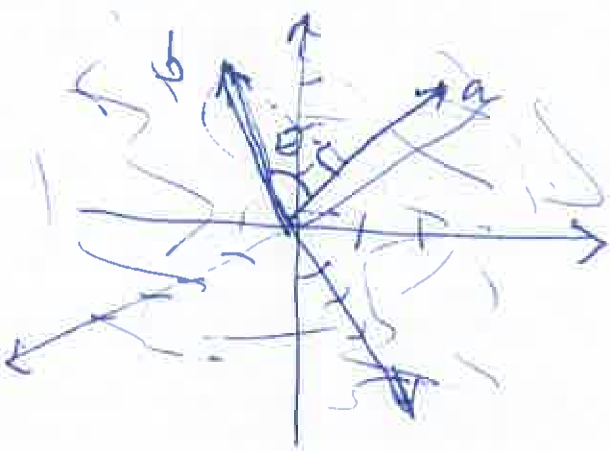
Then $\langle a | b \rangle = 0 + 2 + 2 + 1 = 5$

$\mathrm{proj}_a(b) = \frac{1}{3} \langle a | b \rangle \frac{a}{\|a\|} = \frac{1}{3} \cdot 5 \cdot \frac{1}{3} \langle 0, 2, 2, -1 \rangle$

$= \langle 0, \frac{10}{9}, \frac{10}{9}, \frac{-5}{3} \rangle$

and $\cos(\theta) = \frac{1}{3} \cdot \frac{1}{2} \langle a | b \rangle = \frac{1}{6} \cdot 5 = \frac{5}{6}$

so that $\theta = 0.585685434\ldots$



in four dimensions.

Cofactor formulas for the determinant

For 2x2 matrices Pick a row or column.

I'm going to choose row 1.

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\cdot(-1)^{1+1}\det\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ \cancel{a_{21}} & a_{22} \end{pmatrix}$$

$$+ a_{12}(-1)^{1+2}\det\begin{pmatrix} \cancel{a_{11}} & \cancel{a_{12}} \\ a_{21} & \cancel{a_{22}} \end{pmatrix}$$

$$= a_{11}\det(a_{22}) - a_{12}\det(a_{21})$$

$$= a_{11}a_{22} - a_{12}a_{21}.$$

Example $\det\begin{pmatrix} 2 & 3 \\ 6 & 1 \end{pmatrix} = 2\cdot 1 - 3\cdot 6 = 2 - 18 = -16.$

Cofactor formulas for the determinant

For 3×3 matrices: Pick a row or column

I'm going to choose row 3.

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{31}(-1)^{3+1}\det\begin{pmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}}\end{pmatrix}$$

$$+ a_{32}(-1)^{3+2}\det\begin{pmatrix} a_{11} & \cancel{a_{12}} & a_{13} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}}\end{pmatrix}$$

$$+ a_{33}(-1)^{3+3}\det\begin{pmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ a_{21} & a_{22} & \cancel{a_{23}} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}}\end{pmatrix}$$

$$= a_{31}\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23}\end{pmatrix} - a_{32}\det\begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23}\end{pmatrix} + a_{33}\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$$

Example 15 Topic 2

$$\det\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3\end{pmatrix} = 0\cdot(-1)^{3+1}\det\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}$$

$$+ 1\cdot(-1)^{3+2}\det\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix}$$

$$+ 3\cdot(-1)^{3+3}\det\begin{pmatrix} 1 & 2 \\ -1 & 1\end{pmatrix}$$

$$= 0 - (1-(-1)) + 3(1-(-2))$$

$$= -2 + 3\cdot 3 = 7$$

For 4x4 matrices Pick a row or column.
I'm going to choose column 4.

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = a_{14}(-1)^{1+4}\det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$+ a_{24}(-1)^{2+4}\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$+ a_{34}\cdot(-1)^{3+4}\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$+ a_{44}\cdot(-1)^{4+4}\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Topic 2 Example 16

$$\det\begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix} = 1\cdot(-1)^{1+4}\det\begin{pmatrix} 3 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & -4 & 2 \end{pmatrix} + 0 + 0$$

$$+ 4\cdot(-1)^{4+4}\cdot\det\begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= (-1)\cdot\left(1\cdot(-1)^{2+1}\det\begin{pmatrix} 2 & 2 \\ -4 & 2 \end{pmatrix} + 0 + 1\cdot(-1)^{2+3}\det\begin{pmatrix} 3 & 2 \\ 0 & -4 \end{pmatrix}\right)$$

$$+ (+4)\left(1\cdot(-1)^{3+1}\det\begin{pmatrix} -2 & 0 \\ 2 & 2 \end{pmatrix} + 0 + 1\cdot(-1)^{3+3}\det\begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}\right)$$