10.5 Isotropy and nondegeneracy

Let $W \subseteq V$ be a subspace of V. The *orthogonal to W* is

$$W^\perp = \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w\rangle = 0\}.$$

The subspace W is *nonisotropic* if $W \cap W^\perp = 0$.

Proposition 10.3. *A sesquilinear form $\langle,\rangle\colon V \times V \to \mathbb{F}$ satisfies*

(no isotropic vectors condition) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

if and only if it satisfies

(no isotropic subspaces condition) If W is a subspace of V then $W \cap W^\perp = 0$.

Remark 10.4. Let $V = \mathbb{C}\text{-span}\{e_1, e_2\}$ with symmetric bilinear form $\langle,\rangle\colon V \times V \to \mathbb{C}$ with Gram matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{in the basis } \{e_1, e_2\}.$$

This form has isotropic vectors since $\langle e_1, e_1\rangle = 0$. The dual basis to $\{e_1, e_2\}$ is the basis $\{e_2, e_1\}$. Letting

$$\begin{array}{l} b_1 = \frac{1}{\sqrt{2}}(e_1 + e_2), \\ b_2 = \frac{i}{\sqrt{2}}(e_1 - e_2), \end{array} \qquad \text{then the Gram matrix is} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with respect to the basis $\{b_1, b_2\}$ and $b_1 + ib_2$ is an isotropic vector.

10.6 Nondegeneracy and dual bases

Let V be a $\mathbb{F}$-vector space with a sesquilinear form $\langle,\rangle\colon V \times V \to \mathbb{F}$. The form $\langle,\rangle$ is *nondegenerate* if it satisfies

$$\text{if } v \in V \text{ and } v \neq 0 \text{ then there exists } w \in V \text{ such that } \langle v, w\rangle \neq 0.$$

An alternative way of stating this condition is to say $V \cap V^\perp = 0$. Another alternative is to say that the map

$$\begin{array}{ccc} V & \to & V^* \\ v & \mapsto & \varphi_v \end{array} \qquad \text{given by} \qquad \begin{array}{cccc} \varphi_v\colon & V & \to & \mathbb{F} \\ & w & \mapsto & \langle v, w\rangle \end{array}$$

is an *injective* linear transformation.

Let $k \in \mathbb{Z}_{>0}$ and assume that $W \subseteq V$ is a subspace of V with $\dim(W) = k$. Let $(w_1, \ldots, w_k)$ be a basis of W. A *dual basis to $(w_1, \ldots, w_k)$ with respect to $\langle,\rangle$* is a basis $(w^1, \ldots, w^k)$ of W such that

$$\text{if } i, j \in \{1, \ldots, k\} \text{ then } \quad \langle w^i, w_j\rangle = \delta_{ij}.$$

Proposition 10.5. *Let V be a vector space with a sesquilinear form $\langle,\rangle\colon V \times V \to \mathbb{F}$. Let $W \subseteq V$ be a subspace of V. Assume W is finite dimensional, that $(w_1, \ldots, w_k)$ is a basis of W and that G is the Gram matrix of $\langle,\rangle$ with respect to the basis $\{w_1, \ldots, w_k\}$. The following are equivalent:*

(a) A dual basis to $(w_1, \ldots, w_k)$ exists.

(b) G is invertible.

(c) $W \cap W^\perp = 0$.

(d) The linear transformation

$$\begin{array}{cccc} \Psi_W\colon & W & \to & W^* \\ & v & \longmapsto & \varphi_v \end{array} \qquad \textit{given by} \qquad \varphi_v(w) = \langle v, w\rangle,$$

is an isomorphism.