Topic 4. Example 21. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by

$$S = \{1+2x+5x^2, 1+x+x^2, 1+2x+3x^2\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1(1+2x+5x^2) + c_2(1+x+x^2) + c_3(1+2x+3x^2) = 0$ then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1(1+2x+5x^2) + c_2(1+x+x^2) + c_3(1+2x+3x^2) = 0$.
Then

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ 2c_1 + c_2 + 2c_3 &= 0, \\ 5c_1 + c_2 + 3c_2 &= 0, \end{aligned} \qquad \text{or, equivalently,} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 7 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{(Ex21)}$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or and assignment!), this system has only one solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So S is linearly independent.

Row reduction steps for Example 21.

Multiply both sides of the equation $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 7 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{5} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 7 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{So} \quad \begin{pmatrix} 1 & 1 & 1 \\ 5 & 1 & 2 \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Multiply both sides by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 5 & 1 & 2 \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{So} \quad \begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{4}{3} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{4}{3} \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{4}{5} & \frac{3}{5} \\ 0 & \frac{3}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{4}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \qquad \text{So} \quad \begin{pmatrix} 5 & 1 & 2 \\ 0 & \frac{3}{5} & -\frac{4}{5} \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$\begin{aligned} 5c_1 + c_2 + 2c_3 &= 0, \\ \tfrac{4}{5}c_2 + \tfrac{3}{5}c_3 &= 0 \\ -\tfrac{1}{3}c_3 &= 0. \end{aligned} \qquad \text{The only solution to this system is} \qquad \begin{aligned} c_3 &= 0, \\ c_2 &= 0, \\ c_1 &= 0. \end{aligned}$$