Topic 4. Example 19a. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(2,-1,1),(-6,3,-3)\}. \qquad \text{Is } S \text{ linearly independent?}$$

To show: If $c_1, c_2 \in \mathbb{R}$ and $c_1\,|2,-1,1\rangle + c_2|-6,3,-3\rangle = |0,0,0\rangle$ then $c_1 = 0$, $c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{R}$ and $c_1\,|2,-1,1\rangle + c_2|-6,3,-3\rangle = |0,0,0\rangle$.
Then

$$\begin{aligned} 2c_1 - 6c_2 &= 0, \\ -c_1 + 3c_2 &= 0, \\ c_1 - 3c_2 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2 & -6 \\ -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{Ex19a}$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or and assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

So $c_1 = 0$, $c_2 = 0$ is not the only solution.
So S is not linearly independent.

Topic 4. Example 19b. Let B be the subset of $\mathbb{R}^3$ given by

$$B = \{(2,-1,1),(4,0,2)\}. \qquad \text{Is } B \text{ linearly independent?}$$

To show: If $c_1, c_2 \in \mathbb{R}$ and $c_1\,|2,-1,1\rangle + c_2|4,0,2\rangle = |0,0,0\rangle$ then $c_1 = 0$, $c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{R}$ and $c_1\,|2,-1,1\rangle + c_2|-4,0,2\rangle = |0,0,0\rangle$
Then

$$\begin{aligned} 2c_1 + 4c_2 &= 0, \\ -c_1 + 0c_2 &= 0, \\ c_1 + 2c_2 &= 0, \end{aligned} \qquad \text{or equivalently} \qquad \begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{Ex19b}$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or and assignment!), this system has only one solution $c_1 = 0$, $c_2 = 0$.
So S is linearly independent.