Topic 4. Example 18. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by

$$S = \{1+x+x^2, x^2\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}[x]_{\le 2}.$$

To show: (a) $\mathrm{span}(S) \subseteq \mathbb{R}[x]_{\le 2}$
(b) $\mathbb{R}[x]_{\le 2} \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}[x]_{\le 2}$ and $\mathbb{R}[x]_{\le 2}$ is closed under addition and scalar mutliplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}[x]_{\le 2}$.

(b) To show: $\mathbb{R}[x]_{\le 2} \subseteq \mathrm{span}(S)$.
To show: $\mathbb{R}\text{-span}\{1, x, x^2\} \subseteq \mathrm{span}(S)$.
Since $\mathrm{span}(S)$ is closed under addition and scalar multiplication,
To show: $\{1, x, x^2\} \subseteq \mathrm{span}(S)$.
To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$c_1(1+x+x^2) + c_2x^2 = 1, \quad d_1(1+x+x^2) + d_2x^2 = x, \quad \text{and} \quad r_1(1+x+x^2) + r_2x^2 = x^2.$$

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Multiply both sides by $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get $\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since the top row on the left hand side is all 0 and the top row on the right hand sides is not all 0 then there do not exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{1, x, x^2\} \not\subseteq \mathrm{span}(S)$.
So $\mathbb{R}\text{-span}\{1, x, x^2\} \not\subseteq \mathrm{span}(S)$.
So $\mathbb{R}[x]_{\le 2} \not\subseteq \mathrm{span}(S)$.

So $\mathbb{R}\text{-span}(S) \ne \mathbb{R}[x]_{\le 2}$.