8.6 The 3d-space-time $\mathbb{D}$

The 3*d-space-time*, or the *Hamiltonians*, or the *quaternions*, is the vector space

$$\mathbb{D} = \mathbb{R}\text{-span}\{1, i, j, k\} = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\}$$

with product determined by

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j,$$

and the distributive laws. The 3d-space-time $\mathbb{D}$ is a generalization of 1d-space-time $\mathbb{C}$, where

$$\mathbb{C} = \{xi + t \mid x, t \in \mathbb{R}\} \qquad \text{with product determined by} \quad i^2 = -1 \quad \text{and the distributive laws.}$$

The 3*d-space* $\mathbb{R}^3$ is a subspace of 3d-space-time $\mathbb{D}$,

$$\begin{array}{ll} \mathbb{D} & = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\} \\ \cup| & \\ \mathbb{R}^3 & = \{xi + yj + zk \mid x, y, z \in \mathbb{R}\} \end{array}$$

For $v_1 = x_1 i + y_1 j + z_1 k$ and $v_2 = x_2 i + y_2 j + z_2 k$ in $\mathbb{R}^3$ define

$$\begin{array}{l} \langle v_1 \,|\, v_2 \rangle = x_1 y_1 + y_1 y_2 + z_1 z_2 \qquad \text{and} \\ v_1 \times v_2 = (y_1 z_2 - z_1 y_2)i - (x_1 z_2 - z_1 x_2)j + (x_1 y_2 - y_1 x_2)k. \end{array}$$

Then

$$\begin{aligned} v_1 v_2 &= (x_1 i + y_1 j + z_1 k)(x_2 i + y_2 j + z_2 k) \\ &= -(x_1 x_2 + y_1 y_2 + z_1 z_2) + (x_1 y_2 ij + y_1 x_2 ji) + (x_1 z_2 ik + z_1 x_2 ki) + (y_1 z_2 jk + z_1 y_2 kj) \\ &= -\langle v_1, v_2 \rangle + (x_1 y_2 - y_1 x_2)k - (x_1 z_2 - z_1 x_2)j + (y_1 z_2 - z_1 y_2)i \\ &= -\langle v_1 \,|\, v_2 \rangle + v_1 \times v_2. \end{aligned}$$

This computation shows how the standard inner product and the cross product arise from the mutiplication in 3d-space-time.

Let $v = xi + yj + zk \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

- The *conjugate* of $a = t + v$ is $\bar{a} = t - v$.

If $a = t + v$ then define $\|a\|^2 = a\bar{a}$ so that

$$\|a\|^2 = a\bar{a} = t^2 - tv + tv + \langle v, v \rangle - v \times v = t^2 + \langle v, v \rangle = t^2 + x^2 + y^2 + z^2.$$