## 8.7 Points, lines and planes in $\mathbb{R}^3$

### 8.7.1 Equation of points in $\mathbb{R}^3$

The equations for the point $r_0 = |x_0, y_0, z_0\rangle$ in $\mathbb{R}^3$ are

$$\begin{aligned} x &= x_0, \\ y &= y_0, \\ z &= z_0. \end{aligned} \qquad \qquad \mathit{PICTURE}$$

### 8.7.2 Equations of lines in $\mathbb{R}^3$

Let $r_0 = |x_0, y_0, z_0\rangle \in \mathbb{R}^3$ and $v = |a, b, c\rangle \in \mathbb{R}^3$. The line in $\mathbb{R}^3$ with direction $v$ going through the point $r_0$ is

$$r_0 + \mathbb{R}v = \{r_0 + tv \mid t \in \mathbb{R}\} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\}. \qquad \mathit{PICTURE}$$

The points in the line are the $|x, y, z\rangle$ in $\mathbb{R}^3$ such that

$$\begin{aligned} x &= x_0 + ta, \\ y &= y_0 + tb, \\ z &= z_0 + tc, \end{aligned} \qquad \text{with } t \in \mathbb{R}.$$

Solving for $t$ gives that that the points on the line are the $|x, y, z\rangle$ in $\mathbb{R}^3$ which satisfy the equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

### 8.7.3 Equations of planes in $\mathbb{R}^3$

Let $r_0 = |x_0, y_0, z_0\rangle$ in $\mathbb{R}^3$ and let $u = |u_1, u_2, u_3\rangle$ and $v = |v_1, v_2, v_3\rangle$ in $\mathbb{R}^3$. Let

$$n = |a, b, c\rangle \quad \text{be a vector in } \mathbb{R}^3 \text{ such that } \langle n|u\rangle = 0 \text{ and } \langle n|v\rangle = 0.$$

The plane orthogonal to $n$ going through the point $r_0$ is

$$r_0 + n^\perp = \{r = r_0 + su + tv \mid s, t \in \mathbb{R}\} \qquad \mathit{PICTURE}$$

In other words, the points in the plane are $|x, y, z\rangle$ in $\mathbb{R}^3$ such that

$$\begin{aligned} x &= x_0 + su_1 + tv_1, \\ y &= y_0 + su_2 + tv_2, \\ z &= z_0 + su_3 + tv_3, \end{aligned} \qquad \text{with } s, t \in \mathbb{R}.$$

If $r$ is in the plane then $r - r_0$ is orthogonal to $n$ and so the plane is

$$r_0 + \hat{n}^\perp = \{r = (|x, y, z\rangle \in \mathbb{R}^3 \mid \langle r - r_0|n\rangle = 0\}.$$

If $r = r_0 + su + tv$ is in the plane then $\langle r|n\rangle = \langle r_0 + su + tv|n\rangle = \langle r|n\rangle + s\langle u|n\rangle + t\langle v|n\rangle = \langle r_0|n\rangle + 0 + 0$. So

$$r_0 + n^\perp = \{|x, y, z\rangle \in \mathbb{R}^3 \mid ax + by + cz = d\}, \quad \text{where} \quad n = |a, b, c\rangle \quad \text{and} \quad d = \langle r_0|n\rangle.$$

In other words, the points in the plane are the $|x, y, z\rangle$ in $\mathbb{R}^3$ such that

$$ax + by + cz = d, \qquad \text{where } n = |a, b, c\rangle \text{ and } d = ax_0 + by_0 + cz_0.$$