Topic 3. Example 13. Find a vector form for the line of intersection of the two planes $x+3y+2z=6$ and $3x+2y+z=11$.

The points on the intersection of the two planes are the points $|x,y,z\rangle$ that satisfy the system of equations

$$\begin{aligned} 3x+2y-z&=11\\ x+3y+2z&=6. \end{aligned}$$

In matrix form these equations are

$$\begin{pmatrix} 3 & 2 & -1\\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 11\\ 6 \end{pmatrix}.$$

Multiply both sides by $\begin{pmatrix} 0 & 1\\ 1 & -3 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 3 & 2\\ 0 & -7 & -7 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 6\\ -7 \end{pmatrix}.$$

Multiply both sides by $\begin{pmatrix} 1 & 0\\ 0 & -\frac{1}{7} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 3 & 2\\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 6\\ 1 \end{pmatrix}.$$

Multiply both sides by $\begin{pmatrix} 1 & -3\\ 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 & -1\\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 3\\ 1 \end{pmatrix}.$$

So

$$\begin{aligned} x-z&=3,\\ y+z&=1, \end{aligned} \qquad \text{giving} \qquad \begin{aligned} x&=3+z,\\ y&=1-z,\\ z&=0+z, \end{aligned}$$

where z can be any element of $\mathbb{R}$. So the solutions to these equations are

$$\begin{pmatrix} x\\ y\\ z \end{pmatrix}=\begin{pmatrix} 3\\ 1\\ 0 \end{pmatrix}+\mathrm{span}\left\{\begin{pmatrix} 1\\ -1\\ 1 \end{pmatrix}\right\} \qquad \text{which is the line} \quad \{|3,1,0)+t|1,-1,1| \mid t\in\mathbb{R}\}$$

as the line of intersection of the two planes.