8.4 Determinants and volumes

(Lengths of segments in $\mathbb{R}$) Let P be the segment with vertices $|0\rangle$ and $|u_1\rangle$. Show that

$$(\text{Length of segment } P) = \left|\det\left(u_1\right)\right|. \qquad \textit{PICTURE} \qquad \text{(lengthdet)}$$

(Areas of parallelograms in $\mathbb{R}^2$) Let P be the parallelogram with vertices $(0,0)$, (v_1,v_2), (w_1,w_2) and (v_1+w_1, v_2+w_2). Show that

$$(\text{Area of } P) = \left|\det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}\right|. \qquad \textit{PICTURE} \qquad \text{(areadet)}$$

(Volumes of parallelipipeds $\mathbb{R}^3$) Let P be the parallelipiped with vertices $(0,0,0)$, (u_1,u_2,u_3), (v_1,v_2,v_3), (w_1,w_2,w_3), $(u_1+v_1, u_2+v_2, u_3+v_3)$, $(u_1+w_1, u_2+w_2, u_3+w_3)$, $(v_1+w_1, v_2+w_2, v_3+w_3)$ and $(u_1+v_1+w_1, u_2+v_2+w_2, u_3+v_3+w_3)$. Show that

$$(\text{Volume of paralleipiped } P) = \left|\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right|. \qquad \text{PICTURE} \qquad \text{(volumedet)}$$

Let's expalin why this works. Let E_{ij} be the 3×3 matrix with 1 in the (i,j) entry and 0 elsewhere. Let $1 = E_{11}+E_{22}+E_{33}$,

$$\begin{aligned} s_{ij} &= 1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}, && \text{for } i,j \in \{1,\ldots,3\} \text{ with } i \neq j, \\ x_{ij}(c) &= 1 + cE_{ji}, && \text{for } i,j \in \{1,\ldots,3\} \text{ with } i \neq j \text{ and } c \in \mathbb{R}, \\ d_i(c) &= 1 + (c-1)E_{ii}, && \text{for } i \in \{1,\ldots,3\} \text{ and } c \in \mathbb{R} \text{ with } c \neq 0. \end{aligned}$$

Consider the volume of the parallelipiped as a function of the edge vectors $u = |u_1,u_2,u_3\rangle$, $v = |v_1,v_2,v_3\rangle$ and $w = |w_1,w_2,w_3\rangle$.

$$\mathrm{Vol}(u,v,w) = \mathrm{Vol}\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \mathrm{Vol}(P), \qquad \text{where} \quad P = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Since switching two edges produces the same parallelipiped then

$$\mathrm{Vol}(s_{ij}P) = \mathrm{Vol}(P).$$

Since stretching one of the edges of the parallelipped by a factor of c changes the volume by a factor of c then

$$\mathrm{Vol}(d_{ij}(c)P) = c \cdot \mathrm{Vol}(P). \qquad \textit{PICTURE}$$

Since area of a parallelgram is always (base) $\cdot$ (height) then

$$\mathrm{Vol}(x_{ij}(c)P) = \mathrm{Vol}(P). \qquad \textit{PICTURE}$$

It follows that $\mathrm{Vol}(P)$ can be computed by writing P as a product of elementary matrices and using

$$\mathrm{Vol}(s_{ij}) = 1, \qquad \mathrm{Vol}(d_i(c)) = c, \qquad \mathrm{Vol}(x_{ij}(c)) = 1.$$

Thus the volume of P is the absolute value of the determinant of P,

$$\mathrm{Vol}(P) = |\det(P)|.$$