8.3 Angles, orthogonality and projections

Proposition 8.1. *Let* $x = |x_1, \ldots, x_n\rangle \in \mathbb{R}^n$ *and* $y = |y_1, \ldots, y_n\rangle \in \mathbb{R}^n$.
(a) (Cauchy-Schwarz) $|\langle x \,|\, y\rangle| \le \|x\| \cdot \|y\|$.
(b) (Triangle inequality) $\|x + y\| \le \|x\| + \|y\|$.

Let $x = |x_1, \ldots, x_n\rangle \in \mathbb{R}^n$ and $y = |y_1, \ldots, y_n\rangle \in \mathbb{R}^n$. By Cauchy-Schwarz,

$$-1 \le \frac{\langle x \,|\, y\rangle}{\|x\| \cdot \|y\|} \le 1.$$

Let $\mathbb{R}_{[0,\pi]} = \{\theta \in \mathbb{R} \mid 0 \le \theta \le \pi\}$. The *angle between* x *and* y is $\theta \in \mathbb{R}_{[0,\pi]}$ determined by

$$\cos(\theta) = \frac{\langle x \,|\, y\rangle}{\|x\| \cdot \|y\|}. \qquad \text{GRAPHOFCOS}$$

The vectors x and y are *orthogonal* if $\langle x \,|\, y\rangle = 0$ (so that $\cos(\theta) = 0$ and $\theta = \frac{\pi}{2}$). The *projection of* x *onto* $\mathbb{R}y$ is

$$P_{\mathbb{R}y}(x) = \|x\| \cos(\theta) \frac{1}{\|y\|} y = \frac{\langle x \,|\, y\rangle}{\|y\|^2} y = \frac{\langle x \,|\, y\rangle}{\langle y \,|\, y\rangle} y, \qquad \text{PICTURE}$$

since $P_{\mathbb{R}y}(x)$ is parallel to y of length $\|P_{\mathbb{R}y}(x)\| = \|x\| \cos(\theta)$.

Proof of Proposition 8.1. (a) Let

$$x = |x_1, \ldots, x_n\rangle \in \mathbb{R}^n \qquad \text{and} \qquad y = |y_1, \ldots, y_n\rangle \in \mathbb{R}^n,$$

and assume that $x \ne 0$. Let

$$u = \langle x \,|\, x\rangle y - \langle y \,|\, x\rangle x.$$

Then

$$\begin{aligned} 0 \le \langle u \,|\, u\rangle &= \big\langle \langle x \,|\, x\rangle y - \langle y \,|\, x\rangle x, \langle x \,|\, x\rangle y - \langle y \,|\, x\rangle x \big\rangle \\ &= \langle x \,|\, x\rangle^2 \langle y \,|\, y\rangle - \langle x \,|\, x\rangle\langle y \,|\, x\rangle\langle y \,|\, x\rangle - \langle y \,|\, x\rangle\langle x \,|\, x\rangle\langle x \,|\, y\rangle + \langle y \,|\, x\rangle^2 \langle x \,|\, x\rangle \\ &= \langle x \,|\, x\rangle(\langle x \,|\, x\rangle\langle y \,|\, y\rangle - |\langle y \,|\, x\rangle|^2) \end{aligned}$$

Since $x \ne 0$ then $\langle x \,|\, x\rangle \in \mathbb{R}_{>0}$. Thus,

$$0 \le \langle x \,|\, x\rangle\langle y \,|\, y\rangle - |\langle y \,|\, x\rangle|^2 \qquad \text{and so} \qquad |\langle y \,|\, x\rangle|^2 \le \langle x \,|\, x\rangle\langle y \,|\, y\rangle.$$

If $a, b \in \mathbb{R}_{\ge 0}$ then $a^2 \le b^2$ if and only if $a \le b$. So $|\langle x \,|\, y\rangle| \le \|x\| \cdot \|y\|$.
(b) Assume $x = |x_1, \ldots, x_n\rangle \in \mathbb{R}^n$ and $y = |y_1, \ldots, y_n\rangle \in \mathbb{R}^n$. Then

$$\begin{aligned} \|x + y\|^2 = \langle x \,|\, x\rangle + \langle x \,|\, y\rangle + \langle y \,|\, x\rangle + \langle y \,|\, y\rangle &= \|x\|^2 + |y\|^2 + \langle x \,|\, y\rangle + \langle x \,|\, y\rangle \\ &\le \|x\|^2 + |y\|^2 + 2|\langle x \,|\, y\rangle| \end{aligned}$$

Thus, by Cauchy-Schwarz,

$$\|x + y\|^2 \le \|x\|^2 + |y\|^2 + 2\|x\| \cdot \|y\| = (\|x\| + \|y\|)^2.$$

If $a, b \in \mathbb{R}_{\ge 0}$ then $a^2 \le b^2$ if and only if $a \le b$. So $\|x + y\| \le \|x\| + \|y\|$. □