4.3 Solutions of systems of linear equations

Using matrix multiplication the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

is written in the form

$$Ax = b, \qquad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Define

$$\mathrm{Sol}(Ax = b) = \{x \in \mathbb{F}^n \mid Ax = b\}.$$

The following proposition says that if A is square and invertible then

$$x = A^{-1}Ax = A^{-1}b \qquad \text{is the unique solution to the system of equations } Ax = b,$$

so that $\mathrm{Sol}(Ax = b)$ contains only one element.

Proposition 4.7. *Let $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^m$. If $m = n$ and $A \in GL_n(\mathbb{F})$ then*

$$\mathrm{Sol}(Ax = b) = \{A^{-1}b\}.$$

The following proposition says that $\mathrm{Sol}(Ax = b)$ is the same size as $\ker(A)$.

Proposition 4.8. *Let $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^m$ and assume $\mathrm{Sol}(Ax = b) \neq \emptyset$. Let $p \in \mathrm{Sol}(Ax = b)$. Then*

$$\mathrm{Sol}(Ax = b) = p + \ker(A).$$

The following proposition determines $\mathrm{Sol}(Ax = b)$ explicitly.

Proposition 4.9. *Let $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^n$. Assume $r \in \{1, \ldots, \min(m, n)\}$ and $P \in GL_m(\mathbb{F})$ and $Q \in GL_n(\mathbb{F})$ are such that*

$$A = P1_rQ.$$

(a) If there exists $j \in \{r+1, \ldots, m\}$ such that $(P^{-1}b)_j \neq 0$ then $\mathrm{Sol}(Ax = b) = \emptyset$.

(b) If $(P^{-1}b)_{r+1} = 0, (P^{-1}b)_{r+2} = 0, \ldots, (P^{-1}b)_m = 0$ then $\mathrm{Sol}(Ax = b) \neq \emptyset$ and

$$\mathrm{Sol}(Ax = b) = Q^{-1} \begin{pmatrix} (P^{-1}b)_1 \\ \vdots \\ (P^{-1}b)_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix} | \\ q_{r+1} \\ | \end{pmatrix}, \ldots, \begin{pmatrix} | \\ q_n \\ | \end{pmatrix} \right\},$$

where $q_1, \ldots, q_n$ are the columns of Q^{-1}.