4.5 Matrix groups: Some proofs

4.5.1 The presentation theorem for S_n

Proposition 4.5. *The symmetric group S_n is presented by generators $s_1, s_2, \ldots, s_{n-1}$ and relations*

$$s_i^2 = 1 \qquad \text{and} \qquad s_js_{j+1}s_j = s_{j+1}s_js_{j+1} \qquad \text{and} \qquad s_ks_\ell = s_\ell s_k, \tag{4.8}$$

for $i, j, k, \ell \in \{1, \ldots, n-1\}$ with $j \neq n-1$ and $k \neq \ell \pm 1$.

Proof.
Generators A: the set of permutation matrices.
Relations A: all products of permutations w_1w_2 given by matrix multiplication.
Generators B: $s_1, \ldots, s_{n-1}$.
Relations B: As given in (4.8).

The proof is accomplished in four steps:

(1) Write generators B in terms of generators A.
(2) Deduce relations B from relations A.
(3) Write generators A in terms of generators B.
(4) Deduce relations A from relations B.

Step 1: Generators B in terms of generators A. This is provided by (4.1).
Step 2: Relations B from relations A. This step is given the following matrix computations:

$$s_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$s_1s_2s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$s_2s_1s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

so that $s_1s_2s_1 = s_2s_1s_2$ and

$$s_1s_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$s_3s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

so that $s_1s_3 = s_3s_1$.

Step 3: Generators A in terms of generators B.
Let $w \in S_n$.

Let $j_1 \in \{1, \ldots, n\}$ be such that $w(j_1, 1) = 1$ and let $w^{(1)} = s_1 s_2 \cdots s_{j_1-1} w$.
Let $j_2 \in \{2, \ldots, n\}$ be such that $w^{(1)}(j_2, 2) = 1$ and let $w^{(2)} = s_2 s_3 \cdots s_{j_2-1}$.
Continue this process to obtain

$$\cdots\ (s_2 s_3 \cdots s_{j_2-1})(s_1 s_2 \cdots s_{j_1-1}) w = 1.$$

Thus

$$w = (s_{j_1-1} \cdots s_2 s_1)(s_{j_2-1} \cdots s_3 s_2) \cdots .$$

The expression for w is a reduced word for w and a subword of the reduced word of the longest element given by

$$(s_{n-1} \cdots s_2 s_1)(s_{n-1} \cdots s_3 s_2) \cdots (s_{n-1} s_{n-2}) s_{n-1} = w_0.$$

Step 4: Relations A from relations B.

$$\begin{aligned} s_i(s_{j-1} \cdots s_2 s_1) &= s_{j-1} \cdots s_{i+2} s_i s_{i+1} s_i s_{i-1} \cdots s_2 s_1, && \text{by the third set of relations in (4.8)}, \\ &= s_{j-1} \cdots s_{i+2} s_{i+1} s_i s_{i+1} s_{i-1} \cdots s_2 s_1, && \text{by the second set of relations in (4.8)}, \\ &= (s_{j-1} \cdots s_{i+2} s_{i+1} s_i s_{i-1} \cdots s_2 s_1) s_i, && \text{by the third set of relations in (4.8)}, \end{aligned}$$

So $s_i w$ can be written in normal form. By Step 3, w_1 can be written as a product of simple transpositions, so one simple transposition at a time, $w_1 w$ can be written in normal form. □

If

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{then} \quad s_3(s_2 s_3)(s_1 s_2 w) = s_3(s_2 s_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = s_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 1,$$

so that $w = (s_2 s_1)(s_3 s_2) s_3$.

4.5.2 The presentation theorem for $GL_n(\mathbb{F})$

Theorem 4.6. *The group $GL_n(\mathbb{F})$ is presented by generators*

$$y_i(c), \quad h_j(d), \quad x_{k\ell}(c), \qquad \text{for} \qquad \begin{matrix} c \in \mathbb{F}, d_1, \ldots, d_n \in \mathbb{F}^\times, \\ i \in \{1, \ldots, n-1\}, j \in \{1, \ldots, n\} \\ k, \ell \in \{1, \ldots, n\} \text{ with } k < \ell. \end{matrix}$$

with the following relations:

- *The* reflection relation *is*

$$y_i(c_1) y_i(c_2) = \begin{cases} y_i(c_1 + c_2^{-1}) h_i(c_2) h_{i+1}(-c_2^{-1}) x_{i,i+1}(c_2^{-1}), & \text{if } c_2 \neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2 = 0. \end{cases} \tag{4.9}$$

- *The* building relation *is*

$$y_i(c_1) y_{i+1}(c_2) y_i(c_3) = y_{i+1}(c_3) y_i(c_1 c_3 + c_2) y_{i+1}(c_1). \tag{4.10}$$

- *The* x-interchange relations *are*

$$x_{ij}(c_1)x_{ij}(c_2) = x_{ij}(c_1+c_2),$$
$$x_{ij}(c_1)x_{ik}(c_2) = x_{ik}(c_2)x_{ij}(c_1), \qquad x_{ik}(c_1)x_{jk}(c_2) = x_{jk}(c_2)x_{ik}(c_1),$$
$$x_{ij}(c_1)x_{jk}(c_2) = x_{jk}(c_2)x_{ij}(c_1)x_{ik}(c_1c_2), \qquad x_{jk}(c_1)x_{ij}(c_2) = x_{ij}(c_2)x_{jk}(c_1)x_{ik}(-c_1c_2),$$

  where $i<j<k$.

- *Letting* $h(d_1,\ldots,d_n) = h_1(d_1)\cdots h_n(d_n)$, *the* h-past-y relation *is*

$$h(d_1,\ldots d_n)y_i(c) = y_i(cd_id_{i+1}^{-1})h(d_1,\ldots,d_{i-1},d_{i+1},d_i,d_{i+2},\ldots,d_n). \tag{4.11}$$

- *Letting* $h(d_1,\ldots,d_n) = h_1(d_1)\cdots h_n(d_n)$, *the* h-past-x relation *is*

$$h(d_1,\ldots,d_n)x_{ij}(c) = x_{ij}(cd_id_j^{-1})h(d_1,\ldots,d_n). \tag{4.12}$$

- *The* x-past-y relations *are*

$$\begin{gathered} x_{i,i+1}(c_1)y_i(c_2) = y_i(c_1+c_2)x_{i,i+1}(0), \\ x_{ik}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1c_2)x_{i,k+1}(c_1), \qquad x_{i,k+1}(c_1)y_k(c_2) = y_k(c_2)x_{ik}(c_1), \\ x_{ij}(c_1)y_i(c_2) = y_i(c_2)x_{i+1,j}(c_1), \qquad x_{i+1,j}(c_1)y_i(c_2) = y_i(c_2)x_{ij}(c_1)x_{i+1,j}(-c_1c_2), \end{gathered} \tag{4.13}$$

  where $i<k$ *and* $i+1<j$.

Proof. The proof of this result provides a way of writing an invertible matrix g in a "normal form" as a product of elementary matrices by the following "row reduction" algorithm.

Let $g \in GL_n(\mathbb{F})$.

Let $j_1 \in \{1,2,\ldots,n\}$ be maximal such that $g(j_1,1)\neq 0$. If $j_1=1$ then let $g^{(1)}=g$. If $j_1\neq 1$ then let

$$g^{(1)} = y_1\left(\tfrac{g(1,1)}{g(j_1,1)}\right)^{-1} y_2\left(\tfrac{g(1,2)}{g(j_1,1)}\right)^{-1}\cdots y_{j_1-1}\left(\tfrac{g(j_1-1,1)}{g(j_1,1)}\right)^{-1} g.$$

Now let $j_2\in\{2,\ldots,n\}$ be maximal such that $g^{(1)}(j_2,2)\neq 0$. If $j_2=2$ then let $g^{(2)}=g^{(1)}$. If $j_2\neq 2$ then let

$$g^{(2)} = y_2\left(\tfrac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)}\right)^{-1} y_3\left(\tfrac{g^{(1)}(3,2)}{g^{(1)}(j_2,2)}\right)^{-1}\cdots y_{j_2-1}\left(\tfrac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)}\right)^{-1} g^{(1)}.$$

Continuing this process will produce $g^{(n)}$ which has the property that

the first nonzero entry in row $j+1$ is to the right of the first nonzero entry in row j.

In particular, if g is invertible then $g^{(n)}$ will be upper triangular.

Let $b=g^{(n)}$. Then

$$\begin{aligned} g = \cdots &(y_{j_2-1}\left(\tfrac{g^{(1)}(j_2-1,2)}{g^{(1)}(j_2,2)}\right)\cdots y_3\left(\tfrac{g^{(1)}(3,2)}{g^{(1)}(j_1,2)}\right) y_2\left(\tfrac{g^{(1)}(2,2)}{g^{(1)}(j_2,2)}\right)) \\ &\cdot (y_{j_1-1}\left(\tfrac{g(j_1-1,1)}{g(j_1,1)}\right)\cdots y_2\left(\tfrac{g(2,1)}{g(j_1,1)}\right) y_1\left(\tfrac{g(1,1)}{g(j_1,1)}\right))\cdot b \end{aligned}$$

Checking the relations: Recall that

$$y_i(c) = x_{i,i+1}(c)s_{i,i+1} = \begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}c & 1\\ 1 & 0\end{pmatrix}.$$

The reflection relations and the building relations are the relations for rearranging ys.

Proof of the reflection equation:

If $c_1 \neq 0$ and $c_2 \neq 0$ then

$$\begin{aligned} y_1(c_1)y_1(c_2) &= \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1c_2+1 & c_1 \\ c_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_1+c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 0 & -c_2^{-1} \end{pmatrix} = \begin{pmatrix} c_1+c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 0 \\ 0 & -c_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & c_2^{-1} \\ 0 & 1 \end{pmatrix} \\ &= y_1(c_1+c_2^{-1})h_1(c_2)h_2(-c_2^{-1})x_{12}(c_2^{-1}). \end{aligned}$$

If $c_2 = 0$ then

$$y_1(c_1)y_1(0) = \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} = x_{12}(c_1).$$

Proof of the building relation:

$$\begin{aligned} \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} c_1c_3+c_2 & 1 & 0 \\ c_3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1c_3+c_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

The computation for the proof of the first x-interchange relation is:

$$\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c_1+c_2 \\ 0 & 1 \end{pmatrix}$$

The key computation for the proof of the h-past-y relation is:

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cd_1 & d_1 \\ d_2 & 0 \end{pmatrix} = \begin{pmatrix} cd_1d_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix}$$

Key computations for the proof of the x-past-y relations are:

$$\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1+c_2 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1c_2 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_1c_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & c_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & c_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c_1c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

□