2 Row reduction for $M_{m\times n}(\mathbb{F})$

Let $n \in \mathbb{Z}_{>0}$ and let E_{ij} be the matrix which has 1 in the (i,j) entry and all other entries 0.

Let $\mathbb{F}^\times = \{d \in \mathbb{F} \mid d \neq 0\}$. Let $h(c_1,\ldots,c_n) = c_1E_{11} + \cdots c_nE_{nn}$ for $c_1,\ldots,c_n \in \mathbb{F}^\times$.

$$h(c_1,\ldots,c_n) = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} \qquad \text{with} \qquad h(c_1,\ldots,c_n)^{-1} = \begin{pmatrix} c_1^{-1} & & \\ & \ddots & \\ & & c_n^{-1} \end{pmatrix}.$$

The *elementary diagonal matrices* are $h_i(c) = 1 + (-1+c)E_{ii}$ for $i \in \{1,\ldots,n\}$ and $c \in \mathbb{F}^\times$.

$$h_i(c) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \qquad \text{with} \qquad h_i(c)^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c^{-1} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

The *root matrices* are $x_{ij}(c) = 1 + cE_{ij}$ for $c \in \mathbb{F}$ and $i,j \in \{1,\ldots,n\}$ with $i<j$.

$$x_{ij}(c) = \begin{pmatrix} 1 & & & & \\ & \ddots & & c & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad \text{with} \qquad x_{ij}(c)^{-1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & -c & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

The *row reducers* are $y_i(c) = 1 + (c-1)E_{ii} + E_{i,i+1} + E_{i+1,i} - E_{i+1,i+1}$ for $i \in \{1,\ldots,n-1\}$ and $c \in \mathbb{F}$.

$$y_i(c) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & c & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \qquad \text{with} \qquad y_i(c)^{-1} = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & -c & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

The *simple transpositions* are $s_i = y_i(0)$, for $i \in \{1,\ldots,n-1\}$.

$$s_i = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix} \qquad \text{with} \qquad s_i^{-1} = s_i.$$