5.2.2 Determinants of square matrices: the permutation formula

For $A \in M_n(\mathbb{F})$ define

$$D(A) = \sum_{w \in S_n} \det(w) A(1, w(1)) A(2, w(2)) \cdots A(n, w(n)),$$

Proposition 5.4. *Let $A \in M_n(\mathbb{F})$ and let $i, j \in \{1, \ldots, n\}$ with $i < j$.*

(a) If u is a permutation then $D(uA) = \det(u) D(A)$.

(b) If row i and row j of A are equal then $D(A) = 0$.

(c) $D(AB) = D(A)D(B)$.

(d) $D(s_{ij}) = -1$, $D(x_{ij}(c)) = 1$ and $D(h_i(d)) = c$.

Theorem 5.5. *Let $A \in M_n(\mathbb{F})$. Then*

$$\det(A) = \sum_{w \in S_n} \det(w) A(1, w(1)) A(2, w(2) \cdots A(n, w(n)).$$

5.3 Laplace expansion

Let $J \subseteq \{1, \ldots, n\}$ with $|J| = k$. Write

$$\begin{matrix} J = \{j_1, \ldots, j_k\} \\ J^c = \{\ell_1, \ldots, \ell_{n-k}\} \end{matrix} \qquad \text{where} \qquad \begin{matrix} j_1 < \cdots < j_k \ \text{ and} \\ \ell_1 < \cdots < \ell_{n-k}, \end{matrix}$$

and define a permutation u_J by

$$u_J(r) = \begin{cases} j_r, & \text{if } r \in \{1, \ldots, k\}, \\ \ell_{r-k}, & \text{if } s \in \{k+1, \ldots, n\}. \end{cases}$$

Theorem 5.6. *Let $A \in M_n(\mathbb{F})$.*

(a) (General Laplace expansion) Let $K, L \subseteq \{1, \ldots, n\}$ with $|K| = |L| = k$. Then

$$\sum_{\substack{J \subseteq \mathbb{Z}_{[1,n]} \\ |J| = k}} \det(u_J) \det(A_{K,J}) \det(A^{(L,J)}) = \begin{cases} \det(u_K) \det(A), & \text{if } K = L, \\ 0, & \text{if } K \neq L. \end{cases}$$

where W^J is a set of coset representatives of cosets of S_n / W_J, $A_{K,J}$ is the submatrix of A consisting of entries of A in rows indexed by the elements of K and the entries in columns indexed by J and $A^{(J,L)}$ is the matrix obtained from A by removing the rows indexed by K and removing the columns indexed by elements of L.

(b) (Laplace expansion on the kth row). Let $k, \ell \in \{1, \ldots, n\}$.

$$\sum_{j=1}^{n} (-1)^{k+j} A(k, j) \det(A^{(j;\ell)}) = \begin{cases} \det(A), & \text{if } k = \ell, \\ 0, & \text{if } k \neq \ell. \end{cases}$$