8 Flag varieties

8.0.1 Bruhat decomposition

Theorem 8.1. *Let $G = GL_n(\mathbb{F})$ and B be the subgroup of upper triangular matrices. Then*

$$G = \bigsqcup_{w\in S_n} BwB, \qquad \text{with} \qquad BwB = \bigsqcup_{c_1,\ldots c_\ell\in\mathbb{F}} y_{i_1}(c_1)\cdots y_{i_\ell}(c_\ell)B$$

if $w = s_{i_1}\cdots s_{i_\ell}$ with $\ell = \ell(w)$.

8.0.2 The Bruhat decomposition for $n = 3$

For $c \in \mathbb{F}_q$ define

$$y_1(c) = \begin{pmatrix} c & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad y_2(c) = \begin{pmatrix} 1 & 0 & 0\\ 0 & c & 1\\ 0 & 1 & 0\end{pmatrix},$$

$$s_1 = y_1(0) = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad s_2 = y_2(0) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix},$$

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}, \qquad x_{13}(c) = \begin{pmatrix} 1 & 0 & c\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and for $d \in \mathbb{F}_q^\times$ define

$$h_1(d) = \begin{pmatrix} d & 0 & 0\\ 0 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}, \qquad h_2(d) = \begin{pmatrix} 1 & 0 & 0\\ 0 & d & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad h_3(d) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & d\end{pmatrix}.$$

Then, coset representatives of cosets of B in the double cosets in $B\backslash G/B$

$$\begin{aligned}
B1B &= \{B\},\\
Bs_1B &= \{y_1(c)B \mid c\in\mathbb{F}_q^\times\},\\
Bs_2B &= \{y_2(c)B \mid c\in\mathbb{F}_q^\times\},\\
Bs_1s_2B &= \{y_1(c_1)y_2(c_2)B \mid c_1,c_2\in\mathbb{F}_q^\times\},\\
Bs_2s_1B &= \{y_2(c_1)y_1(c_2)B \mid c_1,c_2\in\mathbb{F}_q^\times\},\\
Bs_1s_2s_1B &= \{y_1(c_1)y_2(c_2)y_1(c_1)B \mid c_1,c_2,c_3\in\mathbb{F}_q^\times\},
\end{aligned}$$

and

$$\begin{aligned}
B &= \{h_1(d_1)h_2(d_2)h_3(d_3)x_{23}(c_3)x_{13}(c_2)x_{12}(c_1) \mid c_1c_2c_3\in\mathbb{F}^q, d_1,d_2,d_3\in\mathbb{F}_q^\times\}\\
&= \left\{ \begin{pmatrix} d_1 & d_1c_1 & d_1c_2\\ 0 & d_2 & d_2c_3\\ 0 & 0 & d_3\end{pmatrix} \;\middle|\; c_1,c_2,c_3\in\mathbb{F}^q, d_1,d_2,d_3\in\mathbb{F}_q^\times \right\},
\end{aligned}$$

so that $\mathrm{Card}(B) = (q-1)^3q^3$ (or, in general, $\mathrm{Card}(B) = (q-1)^n q^{\frac{1}{2}n(n-1)}$).