1.2 The complex numbers $\mathbb{C}$

The *complex numbers* is the $\mathbb{R}$-algebra

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\} \qquad \text{with} \quad i^2 = -1,$$

so that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned} \qquad \text{and} \qquad \begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + i(x_1y_2 + x_2y_1) + i^2y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

The *complex conjugation*, or *Galois automorphism*, is the $\mathbb{R}$-linear map

$$\overline{} \colon \mathbb{C} \to \mathbb{C} \qquad \text{given by} \quad \overline{x + iy} = x - iy.$$

The *norm*, or *length function*, on $\mathbb{C}$ is the function

$$|\ | \colon \mathbb{C} \to \mathbb{R}_{\geq 0} \qquad \text{given by} \quad |x + iy| = \sqrt{x^2 + y^2}.$$

The *Hermitian form*, or *inner product*, on $\mathbb{C}$ is

$$\langle, \rangle \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C} \qquad \text{given by} \quad \langle z_1, z_2 \rangle = z_1 \overline{z_2}.$$

The *Cartesian form* of a complex number $z = x + iy$, the *polar form* $z = re^{i\theta}$, the *real part* $\mathrm{Re}(z)$, the *imaginary part* $\mathrm{Im}(z)$, the *modulus* $|z|$, and the *argument* $\mathrm{Arg}(z)$, are related by

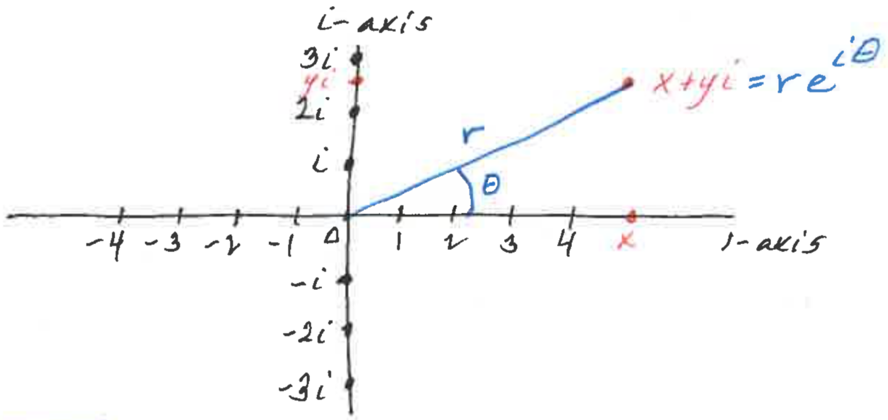
$$z = x + iy = \mathrm{Re}(z) + i\,\mathrm{Im}(z) = re^{i\theta} = |z|e^{i\mathrm{Arg}(z)}, \qquad \text{and}$$

$$z = x + iy, \quad \overline{z} = x - iy, \quad x = \tfrac{1}{2}(z + \overline{z}), \quad y = -i\tfrac{1}{2}(z - \overline{z})$$

so that

$$\mathrm{Re}(z) = \tfrac{1}{2}(z + \bar{z}) = r\cos\theta, \qquad \mathrm{Im}(z) = \tfrac{1}{2i}(z - \bar{z}) = r\sin\theta,$$

$$|z| = \sqrt{x^2 + y^2}, \qquad \mathrm{Arg}(z) = \arctan\left(\tfrac{y}{x}\right).$$



Graphing complex numbers

In particular,

$$re^{i\theta} = r\cos\theta + ir\sin\theta.$$

If $z \in \mathbb{C}$ and $z \neq 0$ then

$$z^{-1} = \frac{1}{|z|^2}\overline{z},$$