7 Vocabulary

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metric
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metric space uniformity
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normal space
norm-absolutey convergent series
nowhere dense set
open ball
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open set (top. space)
operator norm
ordered field
orthogonal complement
orthonormal sequence
orthonormal basis
partition of a set

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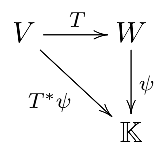
path connected
pointwise convergent
uniformly convergent
poset
product metric space
rational numbers $\mathbb{Q}$
real numbers $\mathbb{R}$
relation
self adjoint operator
separable space
subcover
subset
sup
surjective function
standard metric
subsequence
tolerance set
topology
topological space
triangle ineq. (metric)
triangle inequality (norm)
uniform space
uniformly continuous
pointwise convergent
uniformly convergent
unitary operator
unit circle
unit sphere
vector space

adherent point (deprecated)
Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. A *close point*, or *adherent point*, to E is an element $x \in X$ such that if N is a neighborhood of x then $N \cap E \neq \emptyset$.

adjoint
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V and W be normed $\mathbb{K}$-vector spaces. Let $T\colon V \to W$ be a bounded linear operator. The *adjoint of* T is the function

$$T^*\colon W^* \to V^* \qquad \text{given by} \quad (T^*\psi)(v) = (\psi \circ T)(v).$$



adjoint with respect to $\langle,\rangle$
Let V be an $\mathbb{F}$-vector space with a nondegenerate sesquilinear form $\langle,\rangle\colon V \times V \to \mathbb{F}$. Let $f\colon V \to V$ be a linear transformation. The *adjoint of* f *with respect to* $\langle,\rangle$ is the linear transformation $f^*\colon V \to V$ determined by

$$\text{if } x, y \in V \text{ then} \quad \langle f(x), y\rangle = \langle x, f^*(y)\rangle.$$

basis (vector space)
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. A *basis* of V is a subset $B \subseteq V$ such that

(a) $\mathbb{F}\text{-span}(B) = V$,
(b) B is linearly independent,

where
$$\mathbb{F}\text{-span}(B) = \{a_1b_1 + \cdots + a_\ell b_\ell \mid \ell \in \mathbb{Z}_{>0},\ b_1, \ldots, b_\ell \in B,\ a_1, \ldots, a_\ell \in \mathbb{F}\},$$

and B is *linearly independent* if B satisfies

$$\text{if } \ell \in \mathbb{Z}_{>0} \text{ and } b_1, \ldots, b_\ell \in B \text{ and } a_1, \ldots, a_\ell \in \mathbb{F}, \text{ and}$$
$$a_1b_1 + \cdots + a_\ell b_\ell = 0 \quad \text{then} \quad a_1 = 0,\ a_2 = 0,\ \ldots,\ a_\ell = 0.$$

basis (normed vector space)
Let $\mathbb{F}$ be a field and let V be an normed $\mathbb{F}$-vector space with norm $\| \,\| \colon V \to \mathbb{R}_{\geq 0}$. A *basis* of V (as a normed vector space), or *a topological basis of* V, is a subset $B \subseteq V$ such that

(a) $\overline{\mathbb{F}\text{-span}(B)} = V$,
(b) B is linearly independent,

where $\overline{\mathbb{F}\text{-span}(B)}$ is the closure of $\mathbb{F}$-span(B) in V.

bijective function
A *bijective function* is a function $f \colon X \to Y$ such that f is injective and surjective.

B_ϵ
Let (X, d) be a metric space. Let $\epsilon \in \mathbb{E}$, where $\mathbb{E}$ is the tolerance set. The *diagonal of width* ϵ, or ϵ*-diagonal*, is

$$B_\epsilon = \{(y, x) \in X \times X \mid d(x, y) < \epsilon\}.$$

$B_\epsilon(x)$
Let (X, d) be a metric space. Let $x \in X$ and $\epsilon \in \mathbb{E}$, where $\mathbb{E}$ is the tolerance set. The ϵ*-ball at* x, or *open ball of radius* ϵ *at* x, is

$$B_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}.$$

B(V,W)
Let V and W be normed vector spaces. The space of *bounded operators from* V *to* W is

$$B(V, W) = \{\text{linear transformations } T \colon V \to W \mid \|T\| \text{ exists in } \mathbb{R}_{\geq 0}\} \qquad \text{where}$$

$$\|T\| = \sup\left\{\frac{\|Tv\|}{\|v\|} \mid v \in V\right\}.$$

Banach space
A *Banach space* is a normed vector space $(V, \| \,\|)$ which is complete as a metric space with metric

$$d \colon V \times V \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x, y) = \|x - y\|.$$

Bessel's inequality
Let H be a Hilbert space and let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H. *Bessell's inequality* says that

$$\text{If } x \in H \text{ then} \qquad \sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2 \leq \|x\|^2.$$

bounded function
Let X be a set and let (Y, d) be a metric space. A *bounded function* from X to Y is a function $f\colon X \to Y$ such that

$$f(X) \text{ is a bounded subset of } Y.$$

bounded linear operator
Let V and W be normed vector spaces. A *bounded linear operator from V to W* is a linear transformation $T\colon V \to W$ such that $\|T\|$ exists in $\mathbb{R}_{\geq 0}$, where

$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \right\}.$$

bounded set
The *tolerance set* is $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$. Let (X, d) be a metric space. A *bounded set* in X is a subset $A \subseteq X$ such that

$$\text{there exists } x \in X \text{ and } \epsilon \in \mathbb{E} \quad \text{such that} \quad A \subseteq B_\epsilon(x).$$

C(X)

Cantor set

Cauchy-Schwarz inequality
Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a positive definite Hermitian form $\langle, \rangle\colon V \times V \to \mathbb{R}_{\geq 0}$ and let $\| \; \|\colon V \to \mathbb{R}_{\geq 0}$ be defined by $\|v\|^2 = \langle v, v\rangle$. The *Cauchy-Schwarz inequality* is

$$\text{If } x, y \in V \qquad \text{then} \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

Cauchy sequence
Let (X, d) be a metric space. A *Cauchy sequence* in X is a sequence $(x_1, x_2, \ldots)$ in X such that

$$\text{if } \epsilon \in \mathbb{R}_{>0} \quad \text{then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \quad \text{if } m, n \in \mathbb{Z}_{\geq N} \text{ then } (x_m, x_n) \in B_\epsilon.$$

close point
Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. A *close point to A* is

$$\text{an element } x \in X \text{ such that} \quad \text{if } N \in \mathcal{N}(x) \text{ then } N \cap A \neq \emptyset.$$

cluster point
Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. A *cluster point of* $(x_1, x_2, \ldots)$ is $z \in X$ such that

$$\text{if } \epsilon \in \mathbb{E} \text{ and } \ell \in \mathbb{Z}_{\geq 0} \text{ then there exists } n \in \mathbb{Z}_{\geq \ell} \text{ such that } x_n \in B_\epsilon(z).$$

or, alternatively,

$$\text{there exists a subsequence } (x_{n_1}, x_{n_2}, \ldots) \text{ of } (x_1, x_2, \ldots) \text{ such that} \qquad z = \lim_{k\to\infty} x_{n_k}.$$

limit point
Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. A *limit point of* $(x_1, x_2, \ldots)$ is $z \in X$ such that

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{\geq 0} \text{ such that} \qquad \text{if } n \in \mathbb{Z}_{\geq \ell} \text{ then } x_n \in B_\epsilon(z).$$

or, alternatively, $z = \lim_{n\to\infty} x_n$.

closed set (metric space)
Let (X,d) be a metric space. A *closed set in* X is a subset $W \subseteq X$ such that $\overline{W} = W$ (where $\overline{W}$ is the closure of W in X).

closed set (topological space)
Let $(X,\mathcal{T})$ be a topological space. A *closed set in* X is a subset $A \subseteq X$ such that A^c is open.

closure (metric space)
Let (X,d) be a metric space and let W be a subset of X. The *closure of* W *in* X is

$$\overline{W} = \big\{x \in X \ \big|\ \text{there exists a sequence } (w_1, w_2, \ldots) \text{ in } W \text{ with } \lim_{n\to\infty} w_n = x\big\}.$$

closure (topological space)
Let $(X,\mathcal{T})$ be a topological space and let A be a subset of X. The *closure of* A *in* X is the subset $\overline{A}$ of X such that

(a) $\overline{A}$ is closed in X and $\overline{A} \supseteq A$,
(b) If C is closed in X and $C \supseteq A$ then $C \supseteq \overline{A}$.

compact operator
Let $(V, \|\ \|)V)$ and $(W, \|\ \|_W)$ be Banach spaces. A *compact operator* $T\colon V \to W$ is a bounded linear operator $T\colon V \to W$

$$\begin{array}{l}\text{if } (x_1, x_2, x_3, \ldots) \text{ is a bounded sequence in } V \\ \text{then } (T(x_1), T(x_2), T(x_3), \ldots) \text{ has a cluster point in } W.\end{array}$$

Equivalently, $T\colon V \to W$ is a compact operator if

$$\overline{T(S)} \text{ is compact,} \qquad \text{where} \quad S = \{v \in H \mid \|v\| = 1\}$$

and $\overline{T(S)}$ is the closure of $T(S)$.

compact (sequentially compact)
Let (X, d) be a metric space. Let $A \subseteq X$. The set A is *sequentially compact* if A satisfies

$$\begin{array}{c}\text{if } (a_1, a_2, \ldots) \text{ is a sequence in } A \text{ then}\\ \text{there exists } z \in A \text{ such that} \quad z \text{ is a cluster point of } (a_1, a_2, \ldots).\end{array}$$

(In English: Every infinite sequence in A has a cluster point in A.)

compact (Cauchy compact)
Let (X, d) be a metric space. Let $A \subseteq X$. The set A is *Cauchy compact*, or *complete*, if A satisfies

$$\begin{array}{c}\text{if } (a_1, a_2, \ldots) \text{ is a Cauchy sequence in } A \text{ then}\\ \text{there exists } z \in A \text{ such that} \quad \lim_{n\to\infty} a_n = z.\end{array}$$

(In English: Every Cauchy sequence in A has a limit point in A.)

compact (cover compact)
Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$. The set A is *compact*, or *cover compact*, if A satisfies

$$\begin{array}{c}\text{if } \mathcal{S} \subseteq \mathcal{T} \text{ and } A \subseteq \bigcup_{U\in\mathcal{S}} U \quad \text{then}\\ \text{there exists } \ell \in \mathbb{Z}_{>0} \text{ and } U_1, \ldots, U_\ell \in \mathcal{S} \quad \text{such that} \quad A \subseteq U_1 \cup U_2 \cup \cdots \cup U_\ell.\end{array}$$

(In English: Every open cover of A has a finite subcover.)

compact (ball compact)
Let (X, d) be a metric space and let $A \subseteq X$. The set A is *ball compact*, or *totally bounded*, or *precompact* if A satisfies

if $\epsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that

$$A \subseteq B_\epsilon(x_1) \cup B_\epsilon(x_2) \cup \cdots B_\epsilon(x_\ell).$$

(In English: A can be covered by a finite number of balls of radius ϵ.)

complement
Let X be a set and let $A \subseteq X$. The *complement of A in X* is the set

$$A^c = \{x \in X \mid x \notin A\}.$$

complete (Cauchy compact)
Let (X, d) be a metric space. Let $A \subseteq X$. The set A is *Cauchy compact*, or *complete*, if every Cauchy sequence in A has a limit point in A.

complete space
A *complete space* or *Cauchy compact space* is a metric space X such that every Cauchy sequence in X has a limit point in X.

completion
Let (X, d) be a metric space. The *completion of* (X, d) is a metric space $(\widehat{X}, \hat{d})$ with an isometry

$$\iota\colon X \to \widehat{X} \quad \text{such that} \qquad (\widehat{X}, \hat{d}) \text{ is complete} \qquad \text{and} \qquad \overline{\iota(X)} = \widehat{X},$$

where $\overline{\iota(X)}$ is the closure of the image of ι.

complexnumbers
The *complex numbers* is the $\mathbb{R}$-algebra $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ with $i^2 = -1$ and with *complex conjugation*

$$\begin{matrix} \mathbb{C} & \to & \mathbb{C} \\ c & \mapsto & \overline{c} \end{matrix} \qquad \text{given by} \qquad \overline{a+bi} = a - bi,$$

and *absolute value*

$$\begin{matrix} \mathbb{C} & \to & \mathbb{R}_{\geq 0} \\ c & \mapsto & |c| \end{matrix} \qquad \text{given by} \qquad |c|^2 = c\,\overline{c}.$$

continuous at a point (metric spaces)
Let (X, d_X) and (Y, d_Y) be metric spaces and let $a \in X$. A function $f\colon X \to Y$ is *continuous at* a if f satisfies the condition

$$\lim_{x \to a} f(x) = f(a).$$

continuous at a point (topological spaces)
Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $a \in X$. A function $f\colon X \to Y$ is *continuous at* a if f satisfies the condition

$$\text{if } V \text{ is a neighborhood of } f(a) \text{ in } Y \quad \text{then} \quad f^{-1}(V) \text{ is a neighborhood of } a \text{ in } X.$$

continuous function (topological spaces)
Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces. A *continuous function from* X *to* Y is a function $f\colon X \to Y$ such that

$$\text{if } V \text{ is an open set of } Y \quad \text{then} \quad f^{-1}(V) \text{ is an open set of } X,$$

continuous function (metric spaces)
Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is *continuous* if f satisfies

$$\text{if } a \in X \text{ then} \qquad \lim_{x \to a} f(x) = f(a).$$

contraction
Let (X, d) be a metric space. A *contraction of* X is a function $f\colon X \to X$ such that there exists $\alpha \in \mathbb{R}_{>0}$ such that $\alpha < 1$ and

$$\text{if } x, y \in X \quad \text{then} \quad d(f(x), f(y)) \leq \alpha d(x, y).$$

convergent series

connected space
A *connected space* is a topological space $(X, \mathcal{T})$ such that there do not exist open sets U and V of X such that

$$U \neq \emptyset, \quad V \neq \emptyset, \quad X = U \cup V, \quad \text{and} \quad U \cap V = \emptyset.$$

connected set
Let $(X, \mathcal{T})$ be a topological space. A *connected set in* X is a subset A of X such that there does not exist open sets U and V of X such that

$$A \cap U \neq \emptyset, \quad A \cap V \neq \emptyset, \quad A \subseteq U \cup V, \quad \text{and} \quad (A \cap U) \cap (A \cap V) = \emptyset.$$

connected component
Let $(X, \mathcal{T})$ be a topological space. Define a relation on X by

$$x \sim y \qquad \text{if there exists a connected set } E \subseteq X \text{ such that } x \in E \text{ and } y \in E.$$

Show that $\sim$ is an equivalence relation on X. The *connected components of* X are the equivalence classes with respect to the relation $\sim$.

converges pointwise
Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f \colon X \to C\} \qquad \text{and define} \quad d_\infty \colon F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

Let $(f_1, f_2, \ldots)$ be a sequence in F and let $f \colon X \to C$ be a function. The sequence $(f_1, f_2, \ldots)$ in F *converges pointwise to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\begin{aligned}&\text{if } x \in X \text{ and } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } n \in \mathbb{Z}_{>0} \text{ such that}\\ &\qquad \text{if } n \in \mathbb{Z}_{\geq N} \text{ then } d(f_n(x), f(x)) < \epsilon.\end{aligned}$$

converges uniformly
Let (X, d) and (C, ρ) be metric spaces. Let

$$F = \{\text{functions } f \colon X \to C\} \qquad \text{and define} \quad d_\infty \colon F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{by}$$

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

Let $(f_1, f_2, \ldots)$ be a sequence in F and let $f \colon X \to C$ be a function. The sequence $(f_1, f_2, \ldots)$ in F *converges uniformly to* f if the sequence $(f_1, f_2, \ldots)$ satisfies

$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } n \in \mathbb{Z}_{>0} \text{ such that}\\ &\qquad \text{if } x \in X \text{ and } n \in \mathbb{Z}_{\geq N} \text{ then } d(f_n(x), f(x)) < \epsilon.\end{aligned}$$

dense set
Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The set A is *dense* in X if $\overline{A} = X$.

diameter
Let X be a set and let A be a nonempty subset of X. The *diameter of* A is

$$\mathrm{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

discrete metric
Let X be a set. The *discrete metric* on X is the function

$$d\colon X \times X \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

discrete space
A *discrete space* is a set X with the topology $\mathcal{T}$ equal to the set of all subsets of X.

distance between sets
Let X be a set and let A and B be nonempty subsets of X. The *distance between* A *and* B is

$$d(A, B) = \inf\{d(x, y) \mid x \in A,\ y \in B\}.$$

distance between a point and a set
Let (X, d) be a metric space, let A be a non-empty subset of X and let $x \in X$. The *distance between* x *and* A is

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

direct sum

disconnected space
A *disconnected space* is a topological space $(X, \mathcal{T})$ such that there exists a pair of open sets U and V such that

$$U \neq \emptyset, \quad V \neq \emptyset, \quad U \cup V = X, \quad \text{and} \quad U \cap V = \emptyset.$$

dual space (vector space)
Let $\mathbb{F}$ be a field and let W be an $\mathbb{F}$-vector space. The *dual space* to W is the vector space

$$W^* = \mathrm{Hom}(W, \mathbb{F}) = \{\varphi\colon W \to \mathbb{F} \mid \varphi \text{ is a linear transformation}\}$$

with addition and scalar multiplication given by

$$(\varphi_1 + \varphi_2)(w) = \varphi_1(w) + \varphi_2(w) \qquad \text{and} \qquad (c\varphi)(w) = c \cdot \varphi(w).$$

for $\varphi, \varphi_1, \varphi_2 \in W^*$, $w \in W$ and $c \in \mathbb{F}$.

dual space (normed vector space)
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(V, \| \|)$ be a normed $\mathbb{K}$-vector space.

$$\text{The } \textit{dual space to } V \text{ is} \qquad V^* = B(V, \mathbb{K}),$$

where $B(V, \mathbb{K})$ is the space of bounded linear operators from V to $\mathbb{K}$.

eigenspace
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space, let $T: V \to V$ be a linear transformation and let $\lambda \in \mathbb{F}$. The *λ-eigenspace of $T: V \to V$* is

$$V_\lambda = \{v \in V \mid Tv = \lambda v\}.$$

eigenspectrum
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space, let $T: V \to V$ be a linear transformation. The *eigenspectrum of T* is the set

$$\sigma(T) = \{\lambda \in \mathbb{F} \mid V_\lambda \neq 0\},$$

where V_λ is the λ-eigenspace of T.

eigenvector
Let $\mathbb{F}$ be a field, let V be a $\mathbb{F}$-vector space and let $T: V \to V$ be a linear transformation. An *eigenvector of T* is

$$v \in V \quad \text{such that } v \neq 0 \text{ and} \quad Tv \in \mathbb{F}v,$$

where $\mathbb{F}v = \{cv \mid c \in \mathbb{F}\}$.

emptyset
The *emptyset* $\emptyset$ is the set with no elements.

equivalence class
Let $\sim$ be an equivalence relation on a set S and let $s \in S$. The *equivalence class of s* is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

equivalence relation
An *equivalence relation* on S is a relation $\sim$ on S such that

(a) if $s \in S$ then $s \sim s$,
(b) if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$,
(c) if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.

Euclidean metric
The *Euclidean metric* is the metric on $\mathbb{R}^n$, $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, given by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}.$$

Euclidean space $\mathbb{R}^n$
Euclidean space is the $\mathbb{R}$-vector space $\mathbb{R}^n$ with the positive definite Hermitian form $\langle,\rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle (x_1,\ldots,x_n),(y_1,\ldots,y_n)\rangle = x_1y_1 + \cdots x_ny_n.$$

field
A *field* is a set $\mathbb{F}$ with functions

$$\begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & a+b \end{array} \qquad\text{and}\qquad \begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Fa) If $a,b,c\in\mathbb{F}$ then $(a+b)+c = a+(b+c)$,
(Fb) If $a,b\in\mathbb{F}$ then $a+b=b+a$,
(Fc) There exists $0\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad\text{then}\quad 0+a=a \text{ and } a+0=a,$$

(Fd) If $a\in\mathbb{F}$ then there exists $-a\in\mathbb{F}$ such that $a+(-a)=0$ and $(-a)+a=0$,
(Fe) If $a,b,c\in\mathbb{F}$ then $(ab)c=a(bc)$,
(Ff) If $a,b,c\in\mathbb{F}$ then

$$(a+b)c = ac+bc \qquad\text{and}\qquad c(a+b)=ca+cb,$$

(Fg) There exists $1\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad\text{then}\quad 1\cdot a=a \text{ and } a\cdot 1=a,$$

(Fh) If $a\in\mathbb{F}$ and $a\neq 0$ then there exists $a^{-1}\in\mathbb{F}$ such that $aa^{-1}=1$ and $a^{-1}a=1$,
(Fi) If $a,b\in\mathbb{F}$ then $ab=ba$.

fixed point
A *fixed point of a function* $f\colon X\to X$ is

$$x\in X \qquad\text{such that}\quad f(x)=x.$$

Fourier coefficients

Fourier series

Gram-Schmidt process
Let $(V,\langle\,,\,\rangle)$ be a positive definite Hermitian inner product space. Let $v_1,v_2,\ldots$ be a sequence of linearly independent vectors in V. The *Gram-Schmidt process* is the use of the vectors $v_1,v_2,\ldots$ to construct the vectors $a_1,a_2,\ldots$ in V given by

$$a_1 = \frac{v_1}{\|v_1\|} \qquad\text{and}\qquad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1},a_1\rangle a_1 - \cdots - \langle v_{n+1},a_n\rangle a_n}{\|v_{n+1} - \langle v_{n+1},a_1\rangle a_1 - \cdots - \langle v_{n+1},a_n\rangle a_n\|}.$$

Hausdorff space
A *Hausdorff space* is a topological space $(X, \mathcal{T})$ which satisfies

$$\text{if } x, y \in \text{ and } x \neq y \quad \text{then} \quad \text{there exist open sets } U \text{ and } V \text{ in } X \text{ such that}$$
$$x \in U, \quad y \in V, \quad \text{and} \quad U \cap V = \emptyset.$$

Hilbert space
A *Hilbert space* is a positive definite Hermitian inner product space $(V, \langle,\rangle)$ which is a complete metric space with the metric $d\colon V \times V \to \mathbb{R}_{\geq 0}$ given by

$$d(x, y) = \|x - y\|, \qquad \text{where} \qquad \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

Hölder inequality
Let $q \in \mathbb{R}_{\geq 1}$ and let $p \in \mathbb{R}_{>1} \cup \{\infty\}$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Let $x = (x_1, x_2, \ldots) \in \ell^p$, $y = (y_1, y_2, \ldots) \in \ell^q$ and $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots$. The *Hölder inequality* is

$$|\langle x, y\rangle| \leq \|x\|_p \|y\|_q.$$

homeomeorphism, or **isomorphism of topological spaces**
An *homeomorphism*, or *isomorphism of topological spaces*, is a continuous function $f\colon X \to Y$ such that the inverse function $f^{-1}\colon Y \to X$ exists and is continuous.

inf, or **infimum**, or **greatest lower bound**
Let S be a poset and let E be a subset of S. A *infimum of E in S*, or *greatest lower bound of E in S*, is an element $\inf(E) \in S$ such that

(a) $\inf(E)$ is a lower bound of E in S, and
(b) If $l \in S$ is a lower bound of E in S then $l \leq \inf(E)$.

injective function
Let X and Y be sets. A *injective function* from X to Y is a function $f\colon X \to Y$ such that

$$\text{if } x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \quad \text{then} \quad x_1 = x_2.$$

inner product
Whenever anyone uses this word you should respond, "Do you mean Hermitian or symmetric, or positive definite, or nonisotropic, nondegenerate, or sesquilinear, or just bilinear?"

integers $\mathbb{Z}$

interval

interior point
Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. An *interior point of* E is a element $x \in X$ such that there exists a neighborhood N of x such that $N \subseteq E$.

interior
Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$. The *interior of* E is the subset E° of X such that

(a) E° is open and $E^\circ \subseteq E$,
(b) If U is open and $U \subseteq E$ then $U \subseteq E^\circ$.

inverse function
Let X and Y be sets and let $f \colon X \to Y$ be a function from X to Y. The *inverse function to* f is the function $f^{-1} \colon Y \to X$ such that

$$f^{-1} \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_Y.$$

isometry
Let (X, d_X) and (Y, d_Y) be metric spaces. An *isometry* from X to Y is a function $f \colon X \to Y$ such that

$$\text{if } x, y \in X \quad \text{then} \quad d_Y(f(x), f(y)) = d_X(x, y).$$

The Hilbert space ℓ^2

$$\ell^2 = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } (x_1^2 + x_2^2 + \cdots) < \infty\},$$

with inner product $\langle\,,\,\rangle \colon \ell^2 \times \ell^2 \to \mathbb{R}_{\geq 0}$ given by

$$\langle (x_1, x_2, \ldots), (y_1, y_2, \ldots) \rangle = x_1 y_1 + x_2 y_2 + \cdots.$$

The normed vector spaces ℓ^p
Let $p \in \mathbb{R}_{\geq 1}$.

$$\ell^p = \{(x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \text{ and } \|\vec{x}\|_p < \infty\},$$

where

$$\|(x_1, x_2, \ldots)\|_p = \left(\sum_{i \in \mathbb{Z}_{>0}} |x_i|^p \right)^{1/p}.$$

$L^2(X)$
Let (X,μ) be a measure space.

$$L^2(X) = \{f\colon X \to \mathbb{C} \mid \|f\|_2 \text{ exists in } \mathbb{R}_{\ge 0}\},$$

with inner product $\langle , \rangle\colon L^2(X)\times L^2(X) \to \mathbb{C}$ given by

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}d\mu.$$

$L^p(X)$
Let $p \in \mathbb{R}_{\ge 1}$. Let (X,μ) be a measure space.

$$L^p(X) = \{f\colon X \to \mathbb{C} \mid \|f\|_p \text{ exists in } \mathbb{R}_{\ge 0}\},$$

with norm $\| \ \|_p\colon L^p(X) \to \mathbb{R}_{\ge 0}$ given by

$$\|f\|_p = \Big(\int_X |f(x)|^p d\mu\Big)^{1/p}.$$

length norm
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V be a vector space over $\mathbb{K}$ with a positive definite Hermitian form $\langle , \rangle\colon V\times V \to \mathbb{K}$. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{determined by} \qquad \|v\|^2 = \langle v, v\rangle.$$

linear functional
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V be a $\mathbb{K}$-vector space. A *linear functional* on V is a linear transformation $T\colon V \to \mathbb{K}$.

limit of a sequence
Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. A *limit of the sequence* $(x_1, x_2, \ldots)$ is an element $z \in X$ which satisfies
if $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\ge N}$ then $d(x_n, x) < \epsilon$.

cluster point
Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. A *cluster point of* $(x_1, x_2, \ldots)$ is $z \in X$ such that

$$\text{if } \epsilon \in \mathbb{E} \text{ and } \ell \in \mathbb{Z}_{\ge 0} \text{ then there exists } n \in \mathbb{Z}_{\ge \ell} \text{ such that } x_n \in B_\epsilon(z).$$

or, alternatively,

$$\text{there exists a subsequence } (x_{n_1}, x_{n_2}, \ldots) \text{ of } (x_1, x_2, \ldots) \text{ such that } \quad z = \lim_{k\to\infty} x_{n_k}.$$

limit point
Let (X, d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. A *limit point of* $(x_1, x_2, \ldots)$ is $z \in X$ such that

$$\text{if } \epsilon \in \mathbb{E} \text{ then there exists } \ell \in \mathbb{Z}_{\geq 0} \text{ such that} \quad \text{if } n \in \mathbb{Z}_{\geq \ell} \text{ then } x_n \in B_\epsilon(z).$$

or, alternatively, $z = \lim_{n \to \infty} x_n$.

locally compact
Let $(X, \mathcal{T})$ be a topological space. The space X is *locally compact* if X is Hausdorff and

if $x \in X$ then there exists a neighborhood N of x such that N is cover compact.

metric
Let X be a set. A *metric on* X is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $d(x, x) = 0$,
(b) If $x, y \in X$ and $d(x, y) = 0$ then $x = y$,
(c) If $x, y \in X$ then $d(x, y) = d(y, x)$,
(d) If $x, y, z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.

metric space
A *metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $d(x, x) = 0$,
(b) If $x, y \in X$ and $d(x, y) = 0$ then $x = y$,
(c) If $x, y \in X$ then $d(x, y) = d(y, x)$,
(d) If $x, y, z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.

metric space topology
The *metric space topology on* X is

$$\mathcal{T} = \{U \subseteq X \mid \text{if } x \in U \text{ then there exists } \epsilon \in \mathbb{E} \text{ such that } B_\epsilon(x) \subseteq U\}.$$

metric space uniformity
The *metric space uniformity on* X is

$$\mathcal{E} = \{\text{subsets of } X \times X \text{ which contain an } \epsilon\text{-diagonal}\}.$$

metric subspace
Let (X, d) be a metric space. A *metric subspace* of X is a subset Y of X with metric $d_Y : Y \times Y \to \mathbb{R}_{\geq 0}$ given by $d_Y(y_1, y_2) = d(y_1, y_2)$.

Minkowski inequality
Let $x, y \in \mathbb{R}^n$ or let $x, y \in \ell^p$. The *Minkowski inequality* is

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

neighbourhood of x
Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. A *neighborhood* of x is a subset N of X such that

$$\text{there exists } U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \subseteq N.$$

$\mathcal{N}(x)$
Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$. The *neighborhood filter* of x is

$$\mathcal{N}(x) = \{\text{neighborhoods } N \text{ of } x\}.$$

norm
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V be a $\mathbb{K}$-vector space. A *norm* on V is a function $\|\ \|\colon V \to \mathbb{R}_{\ge 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \le \|x\| + \|y\|$,
(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c|\,\|v\|$,
(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

normed vector space
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A *normed vector space* is a $\mathbb{K}$-vector space V with a function $\|\ \|\colon V \to \mathbb{R}_{\ge 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \le \|x\| + \|y\|$,
(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c|\,\|v\|$,
(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

norm metric
Let $(V, \|\ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d\colon V \times V \to \mathbb{R}_{\ge 0} \qquad \text{given by} \qquad d(x,y) = \|x-y\|.$$

normal space
A *normal space* is a topological space $(X, \mathcal{T})$ which satisfies

$$\begin{array}{l} \text{if } A \text{ and } B \text{ are closed sets in } X \text{ and } A \cap B = \emptyset \\ \text{then} \quad \text{there exist open sets } U \text{ and } V \text{ in } X \text{ such that} \\ \qquad\qquad A \subseteq U, \quad B \subseteq V, \quad \text{and} \quad U \cap V = \emptyset. \end{array}$$

norm-absolutely convergent
Let $(V, \|\ \|)$ be a normed vector space. A *norm-absolutely convergent series* in V is

$$\text{a series } \sum_{n\in\mathbb{Z}_{>0}} a_n \text{ in } V \quad \text{such that} \quad \sum_{n\in\mathbb{Z}_{>0}} \|a_n\| \text{ converges.}$$

nowhere dense set
Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. The set A is *nowhere dense* in X if $(\overline{A})^\circ = \emptyset$.

open ball
Let (X, d) be a metric space. Let $\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}$. An *open ball* is a set

$$B_\epsilon(x) = \{y \in X \mid d(y, x) < \epsilon\}, \qquad \text{with } x \in X \text{ and } \epsilon \in \mathbb{E}.$$

open cover
Let $(X, \mathcal{T})$ be a topological space. An *open cover of* X is a collection $\mathcal{S}$ of open subsets of X such that $X \subseteq \left(\bigcup_{U\in\mathcal{S}} U\right)$.

open set (metric space)
Let (X, d) be a metric space. An *open set* is a subset $U \subseteq X$ such that U^c, the complement of U in X, is closed.

open set (topological space)
Let $(X, \mathcal{T})$ be a topological space. An *open set* is a set $U \in \mathcal{T}$.

operator norm
Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W$ be normed vector spaces and let $T\colon V \to W$ be a linear transformation. The *operator norm* of T is

$$\|T\| = \sup\left\{ \frac{\|Tx\|_W}{\|x\|_V} \mid x \in V \right\}.$$

ordered field
An *ordered field* is a field $\mathbb{F}$ with a total order $\leq$ such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \leq b$ then $a + c \leq b + c$,

(OFb) If $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$ then $ab \geq 0$.

orthogonal complement
Let $(V, \langle\,,\,\rangle)$ be an inner product space and let $W \subseteq V$ be a subspace of V. The *orthogonal complement of* W *in* V is

$$W^\perp = \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w\rangle = 0\}.$$

orthonormal sequence
Let $\mathbb{F}$ be a field and let $(V, \langle\,,\,\rangle)$ be an $\mathbb{F}$-vector space with a sequilinear form. An *orthonormal sequence* in V is a sequence $(b_1, b_2, \ldots)$ in V such that

$$\text{if } i, j \in \mathbb{Z}_{>0} \quad \text{then} \quad \langle b_i, b_j \rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

orthonormal basis
Let H be a separable Hilbert space. An *orthonormal basis of* H is a subset $A \subseteq H$ such that A is countable, A is orthonormal, and $\overline{\mathrm{span}(A)} = H$.

partition of a set
A *partition of a set* S is a collection $\mathcal{P}$ of subsets of S such that

(a) If $s \in S$ then there exists $P \in \mathcal{P}$ such that $s \in P$, and
(b) If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \neq \emptyset$ then $P_1 = P_2$.

path
Let $(X, \mathcal{T})$ be a topological space and let $p \in X$ and $q \in X$. A *path from* p *to* q *in* X is

$$\text{a continuous function } f\colon [0,1] \to X \quad \text{such that} \quad f(0) = p \text{ and } f(1) = q.$$

path connected
A *path connected space* is a topological space $(X, \mathcal{T})$ which satisfies

$$\text{if } p, q \in X \quad \text{then} \quad \text{there exists a path from } p \text{ to } q \text{ in } X.$$

pointwise convergent
Let (X, d_X) and (R, D_R) be metric spaces and let $F = \{\text{functions } f\colon X \to R\}$ A sequence $(f_1, f_2, \ldots)$ in F is *pointwise convergent* if there exists a function $f\colon X \to R$ which satisfies

$$\text{if } x \in X \quad \text{then} \quad \lim_{n\to\infty} d_R(f_n(x), f(x)) = 0.$$

uniformly convergent
Let (X, d_X) and (R, d_R) be metric spaces. Let $F = \{\text{functions } f\colon X \to R\}$ and define

$$d_\infty\colon F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\} \quad \text{by} \quad d_\infty(f, g) = \sup\{d_R(f(x), g(x)) \mid x \in X\}.$$

A sequence $(f_1, f_2, \ldots)$ in F is *uniformly convergent* if there exists a function $f\colon X \to R$ such that the sequence $(f_1, f_2, \ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

poset
A *poset*, or *partially ordered set*, is a set S with a relation $\le$ on S such that

(a) If $x \in A$ then $x \le x$,
(b) If $x, y, z \in S$ and $x \le y$ and $y \le z$ then $x \le z$, and
(c) If $x, y \in S$ and $x \le y$ and $y \le x$ then $x = y$.

product metric space
Let (X, σ) and (Y, ρ) be metric spaces. The *product* of X and Y is the set $X \times Y$ with metric $d\colon (X \times Y) \times (X \times Y) \to \mathbb{R}_{\ge 0}$ given by

$$d((x_1, y_1), (x_2, y_2)) = \sigma(x_1, x_2) + \rho(y_1, y_2).$$

rational numbers $\mathbb{Q}$

real numbers $\mathbb{R}$

relation
A *relation* $\sim$ *on* S is a subset $R_\sim$ of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the subset $R_\sim$ so that

$$R_\sim = \{(s_1, s_2) \in S \times S \mid s_1 \sim s_2\}.$$

self adjoint operator
Let H be a Hilbert space. A *self adjoint operator on* H is

$$\text{a bounded linear operator } T\colon H \to H \quad \text{such that} \quad T = T^*,$$

where T^* is the adjoint of T.

separable space
A *separable space* is a metric space (X, d) such that there exists a subset $A \subseteq X$ such that A is countable and $\overline{A} = X$.

subcover
Let X be a set and let $\mathcal{S}$ be a cover of X. A *subcover* of $\mathcal{S}$ is a

$$\text{subset } \mathcal{U} \subseteq \mathcal{S} \quad \text{such that} \quad X \subseteq \Big(\bigcup_{U \in \mathcal{U}} U\Big).$$

subset
Let X be a set. A *subset* of X is a set A such that

$$\text{if } a \in A \quad \text{then} \quad a \in X.$$

Write $A \subseteq X$ if A is a subset of X.

sup
Let S be a poset and let E be a subset of S. A *supremum*, or*least upper bound of E in S* is an element $\sup(E) \in S$ such that

(a) $\sup(E)$ is a upper bound of E in S, and
(b) If $b \in S$ is a upper bound of E in S then $\sup(E) \le b$.

surjective function
Let X and Y be sets. A *surjective function* from X to Y is a function $f: X \to Y$ such that

$$\text{if } y \in Y \quad \text{then} \quad \text{there exists } x \in X \text{ such that } f(x) = y.$$

standard metric
The *standard metric* is the metric

$$d: \mathbb{C} \times \mathbb{C} \to \mathbb{R}_{\ge 0} \qquad \text{given by} \qquad d(z,w) = |z-w|.$$

subsequence
Let X be a set and let $(x_1, x_2, \ldots)$ be a sequence in X. A *subsequence* of $(x_1, x_2, \ldots)$ is a

$$\text{sequence } (x_{i_1}, x_{i_2}, \ldots) \quad \text{with} \quad i_1 < i_2 < i_3 < \ldots.$$

tolerance set
The *tolerance set*, or *set of tolerances*, is

$$\mathbb{E} = \{10^{-1}, 10^{-2}, \ldots\}.$$

topology
Let X be a set. A *topology on X* is a collection $\mathcal{T}$ of subsets of X such that
(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

topological space
A *topological space* is a set X with a collection $\mathcal{T}$ of subsets of X such that
(a) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,
(b) If $\mathcal{S} \subseteq \mathcal{T}$ then $\left(\bigcup_{U \in \mathcal{S}} U\right) \in \mathcal{T}$,
(c) If $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_\ell \in \mathcal{T}$.

triangle inequality for a function $d\colon X\times X\to\mathbb{R}_{\ge 0}$
Let X be a set and let $d\colon X\times X\to\mathbb{R}_{\ge 0}$ be a function. The function d satisfies the *triangle inequality* if d satisfies

$$\text{if } x,y,z\in X \quad\text{then}\quad d(x,y)\le d(x,z)+d(z,y).$$

triangle inequality for $\|\ \|\colon X\to\mathbb{R}_{\ge 0}$
Let X be a vector space over a field $\mathbb{K}$ and let $\|\ \|\colon X\to\mathbb{R}_{\ge 0}$ be a function. The function $\|\ \|\colon X\to\mathbb{R}_{\ge 0}$ satisfies the *triangle inequality* if $\|\ \|\colon X\to\mathbb{R}_{\ge 0}$ satisfies

$$\text{if } x,y\in X \quad\text{then}\quad \|x+y\|\le\|x\|+\|y\|.$$

uniform space
A *uniform space* is a set X with a collection $\mathcal{E}$ of subsets of $X\times X$ such that

(a) (diagonal condition) If $E\in\mathcal{E}$ then $\Delta(X)\subseteq E$,
(b) (upper ideal) If $E\in\mathcal{E}$ and $D\subseteq X\times X$ and $D\supseteq E$ then $D\in\mathcal{E}$,
(c) (finite intersection) If $\ell\in\mathbb{Z}_{>0}$ and $E_1,E_2,\ldots,E_\ell\in\mathcal{E}$ then $E_1\cap E_2\cap\cdots\cap E_\ell\in\mathcal{E}$,
(d) (symmetry condition) If $E\in\mathcal{E}$ then $\sigma(E)\in\mathcal{E}$,
(e) (triangle condition) If $E\in\mathcal{E}$ then there exists $D\in\mathcal{E}$ such that $D\times_X D\subseteq E$.

uniformly continuous function
Let (X,d_X) and (Y,d_Y) be metric spaces. A *uniformly continuous function* is a function $f\colon X\to Y$ such that

if $\epsilon\in\mathbb{E}$ then there exists $\delta\in\mathbb{E}$ such that
if $x,y\in X$ and $(x,y)\in B_\delta$ then $(f(x),f(y))\in B_\epsilon$.

pointwise convergent
Let (X,d_X) and (R,D_R) be metric spaces and let $F=\{\text{functions } f\colon X\to R\}$ A sequence $(f_1,f_2,\ldots)$ in F is *pointwise convergent* if there exists a function $f\colon X\to R$ which satisfies

$$\text{if } x\in X \quad\text{then}\quad \lim_{n\to\infty} d_R(f_n(x),f(x))=0.$$

uniformly convergent
Let (X,d_X) and (R,d_R) be metric spaces. Let $F=\{\text{functions } f\colon X\to R\}$ and define

$$d_\infty\colon F\times F\to\mathbb{R}_{\ge 0}\cup\{\infty\}\quad\text{by}\quad d_\infty(f,g)=\sup\{d_R(f(x),g(x))\mid x\in X\}.$$

A sequence $(f_1,f_2,\ldots)$ in F is *uniformly convergent* if there exists a function $f\colon X\to R$ such that the sequence $(f_1,f_2,\ldots)$ satisfies

$$\lim_{n\to\infty} d_\infty(f_n,f)=0.$$

unitary operator
Let H be a Hilbert space. A *unitary operator on* H is

$$\text{a bounded linear operator } T\colon H \to H \quad \text{such that} \quad TT^* = T^*T = I,$$

where T^* is the adjoint of T and I is the identity operator on H.

unit circle
The *unit circle* is the set

$$\begin{aligned} S^1 &= \{z \in \mathbb{C} \mid |z| = 1\} \\ &= \{e^{2\pi i\theta} \mid \theta \in \mathbb{R}, 0 \le \theta < 2\pi\} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}. \end{aligned}$$

unit sphere
Let $(V, \| \ \|)$ be a normed vector space. The *unit sphere in* V is

$$S = \{v \in V \mid \|v\| = 1\}.$$

vector space
Let $\mathbb{F}$ be a field. A $\mathbb{F}$-*vector space* is a set V with functions

$$\begin{array}{ccc} V \times V & \to & V \\ (v_1, v_2) & \mapsto & v_1 + v_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{F} \times V & \to & V \\ (c, v) & \mapsto & cv \end{array}$$

(*addition* and *scalar multiplication*) such that

(a) If $v_1, v_2, v_3 \in V$ then $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$,
(b) There exists $0 \in V$ such that if $v \in V$ then $0 + v = v$ and $v + 0 = v$,
(c) If $v \in V$ then there exists $-v \in V$ such that $v + (-v) = 0$ and $(-v) + v = 0$,
(d) If $v_1, v_2 \in V$ then $v_1 + v_2 = v_2 + v_1$,
(e) If $c \in \mathbb{F}$ and $v_1, v_2 \in V$ then $c(v_1 + v_2) = cv_1 + cv_2$,
(f) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $(c_1 + c_2)v = c_1v + c_2v$,
(g) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $c_1(c_2v) = (c_1c_2)v$,
(h) If $v \in V$ then $1v = v$.