21 Tutorial 2: Building projections in Hilbert spaces

Learn to prove the following theorems, accurately, efficiently, using quality proof machine, without having to refer to notes. The first step of this process is to work through each and put the reason why each line appears where it appears. The possible reasons are:

(a) (Proof type II) Assume the ifs
(b) (Proof type II) To show the thens
(c) (Rewriting) This is the definition of ________.
(d) (Proof type III) To show something exists, construct it.
(e) (Proof type III) To show the construction is valid.
(f) (Proof type I) Compute the left hand side.
(g) (Proof type I) Compute the right hand side.

Practice each proof so that you can do it efficiently without referring to notes.

21.0.1 Orthonormal sequences and Gram-Schmidt

Let $\mathbb{F}$ be a field and let $(V, \langle\,,\,\rangle)$ be an $\mathbb{F}$-vector space with a sesquilinear form. An *orthonormal sequence* in V is a sequence $(b_1, b_2, \ldots)$ in V such that

$$\text{if } i,j \in \mathbb{Z}_{>0} \quad \text{then} \quad \langle b_i, b_j\rangle = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j.\end{cases}$$

An *orthogonal sequence* in V is a sequence $(b_1, b_2, \ldots)$ in V such that

$$\text{if } i,j \in \mathbb{Z}_{>0} \quad \text{and } i\neq j \quad \text{then} \quad \langle b_i, b_j\rangle = 0.$$

Proposition 21.1. *Let $\mathbb{F}$ be a field and let $(V, \langle,\rangle)$ be an $\mathbb{F}$-vector space with a Hermitian form. An orthonormal sequence $(a_1, a_2, \ldots)$ in V is linearly independent.*

The following theorem guarantees that, in some favourite examples, orthonormal bases exist.

Theorem 21.2. *(Gram-Schmidt) Let V be an $\mathbb{F}$-vector space with a sesquilinear form $\langle,\rangle\colon V\times V \to \mathbb{F}$. Assume that $\langle,\rangle$ is nonisotropic and that $\langle,\rangle$ is Hermitian i.e.,*

(1) (Nonisotropy condition) If $v\in V$ and $\langle v,v\rangle = 0$ then $v=0$, and

(2) (Hermitian condition) If $v_1, v_2 \in V$ then $\langle v_2, v_1\rangle = \overline{\langle v_1, v_2\rangle}$.

Let $p_1, p_2, \ldots$ be a sequence of linear independent elements of V.

(a) Define $b_1 = p_1$ and

$$b_{n+1} = p_{n+1} - \frac{\langle p_{n+1}, b_1\rangle}{\langle b_1, b_1\rangle} b_1 - \cdots - \frac{\langle p_{n+1}, b_n\rangle}{\langle b_n, b_n\rangle} b_n, \qquad \textit{for } n\in\mathbb{Z}_{>0}.$$

Then $(b_1, b_2, \ldots)$ is an orthogonal sequence in V.

(b) Assume that $\mathbb{F}$ is a field in which square roots can be made sense of and that if $v\in V$ and $v\neq 0$ then $\langle v,v\rangle \neq 0$. Define

$$\|v\| = \sqrt{\langle v,v\rangle}, \qquad \textit{for } v\in V.$$

Assume that $n\in\mathbb{Z}_{>0}$ and that $(b_1, \ldots, b_n)$ is an orthogonal basis of V. Define

$$u_1 = \frac{b_1}{\|b_1\|}, \quad \ldots\ , \quad u_n = \frac{b_n}{\|b_n\|}.$$

Then $(u_1, \ldots, u_n)$ is an orthonormal basis of V.

21.0.2 Orthogonal decomposition

Theorem 21.3. *Let V be a Hilbert space. Let W be a subset of V.*

(a) $W^\perp$ is a closed subspace of V.

(b) W is a closed subspace of V if and only if $V = W \oplus W^\perp$.

21.0.3 Building projections in Hilbert spaces

Proposition 21.4. *Let H be a Hilbert space and let W be a closed subspace of H. Let*

$$d(x, W) = \inf\{d(x, w) \mid w \in W\}.$$

(a) If $x \in H$ then there exists a unique $y \in W$ such that $d(x, y) = d(x, W)$.

(b) Define $P_W : H \to H$ by setting $P_W(x) = y$ where y is as in (a). Then P_W is a linear transformation,

$$P_W(x) \in W, \qquad (\mathrm{id} - P_W)(x) \in W^\perp, \qquad \|P_W\| = 1,$$

$$P_W^2 = P_W, \qquad (\mathrm{id} - P_W)^2 = (\mathrm{id} - P_W), \qquad \textit{and} \qquad \mathrm{id} = P_W + (\mathrm{id} - P_W).$$

21.0.4 The span of an orthonormal sequence and its complement

Theorem 21.5. *Let H be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H and let*

$$W = \mathrm{span}\{a_1, a_2, \ldots\}. \qquad \textit{Then} \qquad H = \overline{W} \oplus \overline{W}^\perp.$$

22 Tutorial 2: Solutions

22.1 Orthonormal sequences and Gram-Schmidt

Proposition 22.1. *Let V be an $\mathbb{F}$-vector space with a sesquilinear form. An orthonormal sequence $(b_1, b_2, \ldots)$ in V is linearly independent.*

Proof. Let $(b_1, b_2, \ldots)$ be an orthonormal sequence in V.
To show: $\{b_1, b_2, \ldots\}$ is linearly independent.
To show: If $\ell \in \mathbb{Z}_{>0}$ and $c_1, \ldots, c_\ell \in \mathbb{F}$ and $c_1a_1 + c_2a_2 + \cdots + c_\ell a_\ell = 0$ then $c_1 = 0, c_2 = 0, \ldots, c_\ell = 0$.
Assume $\ell \in \mathbb{Z}_{>0}$ and and $c_1, \ldots, c_\ell \in \mathbb{F}$ and $c_1a_1 + c_2a_2 + \cdots + c_\ell a_\ell = 0$.
To show: If $j \in \{1, \ldots, \ell\}$ then $c_j = 0$.
Assume $j \in \{1, \ldots, \ell\}$.
Then $0 = \langle c_1b_1 + c_2b_2 + \cdots + c_\ell b_\ell, a_j\rangle = c_j\langle b_j, b_j\rangle = \mu_j$.
So $\{b_1, b_2, \ldots\}$ is linearly independent. □

Theorem 22.2. *(Gram-Schmidt) Let V be an $\mathbb{F}$-vector space with a sesquilinear form $\langle , \rangle \colon V \times V \to \mathbb{F}$. Assume that $\langle , \rangle$ is nonisotropic and that $\langle , \rangle$ is Hermitian i.e.,*

(1) (Nonisotropy condition) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$, and

(2) (Hermitian condition) If $v_1, v_2 \in V$ then $\langle v_2, v_1\rangle = \overline{\langle v_1, v_2\rangle}$.

Let $p_1, p_2, \ldots$ be a sequence of linear independent elements of V.

(a) Define $b_1 = p_1$ and

$$b_{n+1} = p_{n+1} - \frac{\langle p_{n+1}, b_1\rangle}{\langle b_1, b_1\rangle}b_1 - \cdots - \frac{\langle p_{n+1}, b_n\rangle}{\langle b_n, b_n\rangle}b_n, \qquad \text{for } n \in \mathbb{Z}_{>0}.$$

Then $(b_1, b_2, \ldots)$ is an orthogonal sequence in V.

(b) Assume that $\mathbb{F}$ is a field in which square roots can be made sense of and that if $v \in V$ and $v \neq 0$ then $\langle v, v\rangle \neq 0$. Define

$$\|v\| = \sqrt{\langle v, v\rangle}, \qquad \text{for } v \in V.$$

Let $(b_1, \ldots, b_n)$ be an orthogonal basis of V. Define

$$u_1 = \frac{b_1}{\|b_1\|}, \quad \ldots , \quad u_n = \frac{b_n}{\|b_n\|}.$$

Then $(u_1, \ldots, u_n)$ is an orthonormal basis of V.

Proof. (a) To show: If $i, j \in \mathbb{Z}_{>0}$ and $i \neq j$ then $\langle b_i, b_j\rangle = 0$.
To show: If $n + 1 \in \mathbb{Z}_{>0}$ and $j \in \{1, \ldots, n\}$ then $\langle b_{n+1}, b_j\rangle = 0$.
The proof is by induction on n.
The base case $n + 1 = 1$ is vacuously true since $\{1, \ldots, n\} = \emptyset$.
Assume $n + 1 \in \mathbb{Z}_{>1}$.
To show: If $j \in \{1, \ldots, n\}$ then $\langle b_{n+1}, b_j\rangle = 0$.
Assume $j \in \{1, \ldots, n\}$.

Then, by induction, $\langle b_i, b_j\rangle = 0$ for $i \in \{1, \ldots, j-1\}$ and $\langle b_k, b_j\rangle = 0$ for $k \in \{j+1, \ldots, n\}$ so that

$$\begin{aligned}
\langle b_{n+1}, b_j\rangle &= \Big\langle p_{n+1} - \frac{\langle p_{n+1}, b_1\rangle}{\langle b_1, b_1\rangle} b_1 - \cdots - \frac{\langle p_{n+1}, b_n\rangle}{\langle b_n, b_n\rangle} b_n, b_j \Big\rangle \\
&= \langle p_{n+1}, b_j\rangle - \frac{\langle p_{n+1}, b_1\rangle}{\langle b_1, b_1\rangle}\langle b_1, b_j\rangle - \cdots - \frac{\langle p_{n+1}, b_n\rangle}{\langle b_n, b_n\rangle}\langle b_n, b_j\rangle \\
&= \langle p_{n+1}, b_j\rangle - \frac{\langle p_{n+1}, b_j\rangle}{\langle b_j, b_j\rangle}\langle b_j, b_j\rangle = \langle p_{n+1}, b_j\rangle - \langle p_{n+1}, b_j\rangle = 0.
\end{aligned}$$

Since $\langle , \rangle$ is sesquilinear then $\langle b_j, b_{n+1}\rangle = \overline{\langle b_{n+1}, b_j\rangle} = \overline{0} = 0$ (where the last equality is true since ???).
So $(b_1, b_2, \ldots)$ is an orthogonal sequence.
(b) Let $i, j \in \{1, \ldots, n\}$.
Since $(b_1, b_2, \ldots, b_n)$ is orthogonal then

$$\langle u_i, u_j\rangle = \Big\langle \frac{b_i}{\|b_i\|}, \frac{b_j}{\|b_j\|} \Big\rangle = \frac{1}{\|b_i\|\overline{\|b_j\|}}\langle b_i, b_j\rangle = 0, \quad \text{if } i \neq j.$$

If $i = j$ then

$$\langle u_i, u_j\rangle = \Big\langle \frac{b_i}{\|b_i\|}, \frac{b_i}{\|b_i\|} \Big\rangle = \frac{1}{\|b_i\|\overline{\|b_i\|}}\langle b_i, b_i\rangle = \frac{\|b_i\|^2}{\|b_i\|\|b_i\|} = 1$$

(where the next to last equality is true since ????) $\square$

22.2 Orthogonal decomposition

Theorem 22.3. *Let V be a Hilbert space. Let W be a subset of V.*
(a) $W^\perp$ is a closed subspace of V.
(b) W is a closed subspace of V if and only if $V = W \oplus W^\perp$.

Proof. (a) Let

$$\begin{array}{rccc} \Phi\colon & V & \to & W^* \\ & v & \mapsto & \Phi_v \end{array} \qquad \text{where} \qquad \begin{array}{rccc} \Phi_v\colon & W & \to & \mathbb{K} \\ & w & \mapsto & \langle v, w\rangle \end{array}$$

Then

$$W^\perp = \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w\rangle = 0\} = \{v \in V \mid \Phi_v = 0\} = \ker\Phi = \Phi^{-1}(\{0\}).$$

Since $\{0\}$ is closed in W^* and Φ is continuous then $W^\perp = \Phi^{-1}(\{0\})$ is closed.
(b) Assume that W is a closed subspace of V.
To show: (ba) $H = \overline{W} + \overline{W}^\perp$.
(bb) $\overline{W} \cap \overline{W}^\perp = 0$.

(ba) If $x \in H$ then $x = P(x) + (x - P(x)) \in \overline{W} + \overline{W}^\perp$.
So $H = \overline{W} + \overline{W}^\perp$.
(bb) If $y \in \overline{W} \cap \overline{W}^\perp$ then $\langle y, y\rangle = 0$, forcing that $y = 0$.
So $\overline{W} \cap \overline{W}^\perp = 0$.

$\square$

22.3 Building projections in Hilbert spaces

Proposition 22.4. *Let H be a Hilbert space and let W be a closed subspace of H. Let*

$$d(x, W) = \inf\{d(x, w) \mid w \in W\}.$$

(a) If $x \in H$ then there exists a unique $y \in W$ such that $d(x, y) = d(x, W)$.

(b) Define $P_W : H \to H$ by setting $P_W(x) = y$ where y is as in (a). Then P_W is a linear transformation,

$$P_W(x) \in W, \qquad (\mathrm{id} - P_W)(x) \in W^\perp, \qquad \|P_W\| = 1,$$

$$P_W^2 = P_W, \qquad (\mathrm{id} - P_W)^2 = (\mathrm{id} - P_W), \qquad \textit{and} \qquad \mathrm{id} = P_W + (\mathrm{id} - P_W).$$

Proof. (a) Assume $x \in H$.
To show: (aa) There exists $y \in W$ such that $d(x, y) = d(x, W)$.
(ab) If $y, y' \in W$ such that $d(x, y) = d(x, y') = d(x, W)$ then $y = y'$.

(aa) Let $\alpha = d(x, W)$
Let $(y_1, y_2, \ldots)$ be a sequence in W such that

$$\lim_{n\to\infty} \|x - y_n\| = \alpha.$$

To show: There exists $y \in W$ such that $\lim_{n\to\infty} y_n = y$.
Since W is closed, To show: There exists $y \in H$ such that $\lim_{n\to\infty} y_n = y$.
Since H is complete, To show: $(y_1, y_2, \ldots)$ is a Cauchy sequence.
To show: If $\epsilon \in \mathbb{R}_{\ge 0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\ge N}$ then $\|y_m - y_n\| < \epsilon$.
Assum $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\ge N}$ then $\|y_m - y_n\| < \epsilon$.
Let

$$\delta = \min\{1, \frac{\epsilon}{8\alpha + 4}\}$$

and let $N \in \mathbb{Z}_{\ge 0}$ such that if $m \in \mathbb{Z}_{\ge N}$ then $|\, \|x - y_m\| - \alpha| < \delta$.
Since $\frac{1}{2}(y_m + y_n) \in W$ then $\|x - \frac{1}{2}(y_m + y_n)\|^2 \ge \alpha^2$ and the parallelogram identity gives that

$$\begin{aligned}
\|y_m - y_n\|^2 &= 2\|x - y_m\|^2 + 2\|x - y_n\|^2 - 4\|x - \tfrac{1}{2}(y_m + y_n)\|^2 \\
&< 2(\alpha + \delta)^2 + 2(\alpha + \delta)^2 - 4\alpha^2 = 8\alpha\delta + 4\delta^2 = \delta(8\alpha + 4\delta) \\
&\le \delta(8\alpha + 4) \le \frac{\epsilon}{(8\alpha + 4)}(8\alpha + 4) = \epsilon.
\end{aligned}$$

So $(y_1, y_2, \ldots)$ is Cauchy.
So $y = \lim_{n\to\infty} y_n$ exists and

$$d(x, y) = \lim_{n\to\infty} d(x, y_n) = \lim_{n\to\infty} \|x - y_n\| = \alpha = d(x, W).$$

(ab) Assume $y, y' \in W$ and $d(x, y) = d(x, y') = d(x, W)$.
Since $\frac{1}{2}(y + y') \in W$ gives that $d(x, \frac{1}{2}(y + y')) \ge d(x, W) = \alpha$, the parallelogram identity gives

$$\begin{aligned}
\|y - y'\|^2 &= 2\|x - y\|^2 + 2\|x - y'\|^2 - 4\|x - \tfrac{1}{2}(y + y')\|^2 \\
&= 2d(x, y)^2 + 2d(x, y')^2 - 4d(x, \tfrac{1}{2}(y + y'))^2 \\
&\le 2\alpha^2 + 2\alpha^2 - 4\alpha^2 = 0.
\end{aligned}$$

So $y = y'$.

Step 3. To show: $x - P_W(x) \in W^\perp$.
Let $y = P_W(x)$.

Let $w \in W$.
By Step 1, the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(\lambda) = \|x - (y + \lambda w)\|^2 = \|x - y\|^2 + |\lambda|^2\|v\|^2 + 2\mathrm{Re}(\langle x - y, \lambda w\rangle)$$

has unique global minimum at $\lambda = 0$.
So $f'(0) = 0$.
So $\mathrm{Re}(\langle x - y, w\rangle) = 0$.
Similarly, $\mathrm{Re}(\langle x - y, iw\rangle) = 0$.
So $\langle x - y, w\rangle = 0 + 0i$.

Step 4. P_W and $\mathrm{id} - P_W$ are linear operators.

Assume $v_1, v_2 \in H$ and $c_1, c_2 \in \mathbb{K}$.
Then $c_1 P_W(v_1) + c_2 P_W(v_2) \in W$ and $c_1 v_1 + c_2 v_2 - (c_1 P_W(v_1) + c_2 P_W(v_2)) \in W^\perp$.
So $P_W(c_1 v_1 + c_2 v_2) = c_1 P_W(v_1) + c_2 P_W(v_2)$.
So P_W is linear.
So P_W and $\mathrm{id} - P_W$ are linear operators.

Step 5. $\|P_W\| = 1$.

Assume $v \in H$ and $v \neq 0$.
Since $\langle P_W(v), (\mathrm{id} - P_W)(v)\rangle = 0$ then the Pythagorean theorem gives

$$\frac{\|P_W(v)\|^2}{\|v\|^2} \leq \frac{\|P_W(v)\|^2 + \|(\mathrm{id} - P_W)(v)\|^2}{\|v\|^2} = \frac{\|v\|^2}{\|v\|^2} = 1.$$

So $\|P_W\| \leq 1$.
If $w \in W$ then $P_W(w) = w$ so that

$$\|P_W(w)\| = \|w\| \qquad \text{and} \qquad \|P_W\| \geq 1.$$

So $\|P_W\| = 1$.

□

22.4 The span of an orthonormal sequence and its complement

Theorem 22.5. *Let H be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H and let*

$$W = \mathrm{span}\{a_1, a_2, \ldots\}. \qquad \textit{Then} \qquad H = \overline{W} \oplus \overline{W}^\perp.$$

Proof.

Step 1. (Bessel's inequality) If $x \in H$ then $\displaystyle\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2 \leq \|x\|^2$.

Step 2. If $x \in H$ then $P(x) = \displaystyle\sum_{n=1}^{\infty} \langle x, a_n\rangle a_n$ exists in H.

Step 3. If $x \in H$ then $P(x) \in \overline{W}$.

Step 4. If $x \in H$ then $x - P(x) \in \overline{W}^{\perp}$.

Step 5. If $x \in H$ then $P(x) = P_{\overline{W}}(x)$.

Step 1. To show: $\displaystyle\lim_{k\to\infty} \left(\sum_{n=1}^{k} |\langle x, a_n\rangle|^2\right) \le \|x\|^2$.
Assume $k \in \mathbb{Z}_{\ge 0}$. Let

$$x_k = \sum_{n=1}^{k} \langle x, a_n\rangle a_n \qquad \text{so that} \qquad \|x_k\|^2 = \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle} = \sum_{n=1}^{k} |\langle x, a_n\rangle|^2.$$

To show: $\|x_k\|^2 \le \|x\|^2$.
Then

$$\begin{aligned}
\langle x - x_k, x_k\rangle &= \langle x, x_k\rangle - \langle x_k, x_k\rangle \\
&= \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle} - \sum_{n=1}^{k} \langle x, a_n\rangle \overline{\langle x, a_n\rangle} = 0, \qquad \text{and} \\
\|x\|^2 &= \langle x, x\rangle = \langle x_k + (x - x_k), x_k + (x - x_k)\rangle \\
&= \langle x_k, x_k\rangle + \langle x_k, x - x_k\rangle + \langle x - x_k, x_k\rangle + \langle x - x_k, x - x_k\rangle \\
&= \|x_k\|^2 + 0 + 0 + \|x - x_k\|^2.
\end{aligned}$$

So $\|x_k\|^2 \le \|x\|^2$.
So $\displaystyle\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2 = \lim_{k\to\infty} \left(\sum_{n=1}^{k} |\langle x, a_n\rangle|^2\right) = \lim_{k\to\infty} \|x_k\|^2 \le \|x\|^2$.

Step 2. Assume $x \in H$.
Let $x_k = \sum_{n=1}^{k} \langle x, a_n\rangle a_n$.
To show: $\lim_{k\to\infty} x_k$ exists in H.
Since H is complete, we need
To show: $(x_1, x_2, \ldots)$ is a Cauchy sequence in H.
We know that $\|x_k\| = \displaystyle\sum_{n=1}^{k} \langle x, a_n\rangle^2$ so that, by Bessel's inequality,

$$(\|x_1\|, \|x_2\|, \ldots) \text{ is a increasing sequence in } \mathbb{R}_{\ge 0} \text{ bounded by } \|x\|.$$

So $(\|x_1\|, \|x_2\|, \ldots)$ converges.

$$\text{Let} \qquad y = \lim_{k\to\infty} \|x_k\|.$$

To show: If $\epsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\ge N}$ then $\|x_r - x_s\| < \epsilon$.
Assume $\epsilon \in \mathbb{R}_{>0}$.
To show: There exists $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\ge N}$ then $\|x_r - x_s\| < \epsilon$.
Let $N \in \mathbb{Z}_{>0}$ such that if $k \in \mathbb{Z}_{\ge N}$ then $\left|y^2 - \|x_k\|^2\right| < \frac{\epsilon}{2}$.

Assume $r, s \in \mathbb{Z}_{\geq N}$.
To show: $\|x_r - x_s\| < \epsilon$.

$$\begin{aligned}\|x_r - x_s\|^2 &= \left\| \sum_{j=1}^{r} \langle x, a_j\rangle a_j - \sum_{j=1}^{s} \langle x, a_j\rangle a_j \right\|^2 = \left\| \sum_{j=r+1}^{s} \langle x, a_j\rangle a_j \right\|^2 = \sum_{j=r+1}^{s} \langle x, a_j\rangle^2 \\ &= \left|\|x_s\|^2 - \|x_r\|^2\right| = \left|\|x_s\|^2 - y^2 + y^2 - \|x_r\|^2\right| \\ &\leq \left|\|x_s\|^2 - y^2\right| + \left|y^2 - \|x_r\|^2\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

So $(x_1, x_1, \ldots)$ is a Cauchy sequence in H.
So $\lim_{k\to\infty} x_k$ exists in H.
So $\displaystyle\sum_{j=1}^{\infty} \langle x, a_j\rangle a_j$ exists in H.

Step 3. To show: $\displaystyle\sum_{n=1}^{\infty} \langle x, a_n\rangle a_n \in \overline{W}$.
Since

$$x_k = \sum_{j=1}^{k} \langle x, a_j\rangle a_j \quad \text{is an element of} \quad \mathrm{span}\{a_1, a_2, \ldots\} = W$$

then $P(x) = \lim_{k\to\infty} x_k \in \overline{W}$.

Step 4. To show: If $b \in \overline{W}$ then $\langle x - P(x), b\rangle = 0$.
Assume $b \in \overline{W}$.
Let $(b_1, b_2, \ldots)$ be a sequence in W with $\lim_{n\to\infty} b_n = b$.
To show: $\langle x - P(x), b\rangle = 0$.
Using that $\langle x - P(x), \cdot\rangle \colon H \to \mathbb{C}$ is continuous,

$$\langle x - P(x), b\rangle = \langle x - P(x), \lim_{n\to\infty} b_n\rangle = \lim_{n\to\infty} \langle x - P(x), b_n\rangle.$$

Since $b_n \in W$ there exists $\ell \in \mathbb{Z}_{\geq 0}$ and $c_1, c_2, \ldots, c_\ell \in \mathbb{C}$ such that

$$b_n = \sum_{r=1}^{\ell} c_r a_r \qquad \text{and} \qquad \langle x - P(x), b_n\rangle = \sum_{r=1}^{\ell} \overline{c_r} \langle x - P(x), a_r\rangle.$$

Using that $\langle \cdot, a_r\rangle \colon H \to \mathbb{C}$ is continuous and that $\langle x_k, a_r\rangle = \langle x, a_r\rangle$ for $k \geq r$ then

$$\begin{aligned}\langle x - P(x), a_r\rangle &= \langle x, a_r\rangle - \langle P(x), a_r\rangle = \langle x, a_r\rangle - \langle \lim_{k\to\infty} x_k, a_r\rangle \\ &= \langle x, a_r\rangle - \lim_{k\to\infty} \langle x, a_r\rangle = \langle x, a_r\rangle - \langle x, a_r\rangle = 0.\end{aligned}$$

So

$$\langle x - P(x), b_n\rangle = \sum_{j=1}^{\ell} \overline{c_j} \langle x - P(x), a_j\rangle = 0.$$

Thus

$$\langle x - P(x), b\rangle = \lim_{n\to\infty} \langle x - P(x), b_n\rangle = \lim_{n\to\infty} 0 = 0.$$

So $x - P(x) \in \overline{W}^{\perp}$.

Step 5. Since $x - P(x) \in \overline{W}^{\perp}$, $x - P_{\overline{W}} \in \overline{W}^{\perp}$ and $P(x) - P_{\overline{W}} \in \overline{W}$ then

$$\begin{aligned} \|P(x) - P_{\overline{W}}(x)\|^2 &= \|P(x) - x + (x - P_{\overline{W}}(x)), P(x) - P_{\overline{W}}(x)\rangle \\ &= \langle P(x) - x, P(x) - P_{\overline{W}}(x)\rangle + \langle (x - P_{\overline{W}}(x)), P(x) - P_{\overline{W}}(x)\rangle \\ &= 0 + 0 = 0. \end{aligned}$$

So $P(x) = P_{\overline{W}}(x)$. $\square$